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Ambiguity, Randomization and the Timing of Resolution of Uncertainty

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Abstract

The classic framework of Anscombe & Aumann (1963) for decision-making under uncertainty postulates both a primary source of uncertainty and an auxiliary and stochastically independent randomization device. It also imposes a specific timing of resolution of uncertainty as the primary source resolves prior to the randomization device. While this timing is without loss of generality for Subjective Expected Utility, it forbids plausible choice patterns of ambiguity aversion. In this paper, we reverse this timing by assuming that the randomization device resolves first and obtain an axiomatic characterization of Choquet Expected Utility that is dual to that of Schmeidler (1989). In this representation, ambiguity aversion is, somewhat surprisingly, characterized by an aversion to randomizing unambiguous acts on ambiguous events. Moreover, it is quantitatively more pronounced than in Schmeidler's model. Finally, our reversed timing yields the incentive compatibility of the random incentive mechanisms frequently used in experiments for eliciting ambiguous beliefs.

Keywords: Ambiguity Aversion, Randomization, Timing of Resolution of Uncertainty, Choquet Expectation, Slice-Comonotonicity.

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1 Introduction

Ambiguity averse Decision-Makers only dispose of partial information to quantify the uncertainty they face. As exemplified by the Ellsberg (1961) paradox, they express a prefer-

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ence for betting on events of known probability rather than on ones of unknown probability. Since the seminal work of Schmeidler (1989), the framework of Anscombe & Aumann (1963) has often been employed to model the notion of ambiguity aversion in a more formal way. As a distinctive feature, this framework assumes a second source of uncertainty that is infinitely rich and comes equipped with objective probabilities. This richness ensures that each event from the first source has an equivalent in the second source. The objective probabilities of such equivalents then help the Decision-Maker quantify the uncertainty attached to the first source. Furthermore, the second source allows the Decision-Maker to implement randomizations between actions depending on the first source. As noted by Raiffa (1961), such randomizations are especially relevant in the context of ambiguity aversion. Raiffa argued indeed that they help reduce the exposure to ambiguity.

Now, the Anscombe-Aumann (AA) framework[1] also implicitly suggests a specific timing for the resolution of uncertainty where the ambiguous source of uncertainty (i.e. the first source) resolves prior to the randomization device (i.e. the second source). The starting point of the present paper is the observation that these specific timing is sometimes too restrictive to account for ambiguity aversion in a fully satisfactory way. To illustrate, suppose that the ambiguous source of uncertainty consists of an urn containing red and black balls in unknown proportions and that the randomization device takes the form of a fair coin. Next, consider the following four bets:

f	R	B
H	\$10	\$10
T	\$0	\$0

g	R	B
H	\$0	\$0
T	\$10	\$10

h	R	B
H	\$10	\$0
T	\$10	\$0

k	R	B
H	\$0	\$10
T	\$0	\$10

The payoffs induced by each of f and g only depend on the outcome of the randomization device. These two acts can hence be thought of as unambiguous. Meanwhile, h and k induce payoffs only depending on the outcome of the ambiguous source, and can this time be thought of as ambiguous. Symmetry reasons make us assume that f and g are indifferent to each other, and likewise for h and k. In this context, ambiguity aversion typically means a preference for each of f and g over each of h and k. Consider now the following bets:

l	R	B
H	\$10	\$0
T	\$0	\$10

l'	R	B
H	\$10	\$1
T	\$0	\$9

In the AA framework, actions are modeled as functions from the ambiguous state space to the set of probability distributions on the outcomes; these will be referred to as AA acts. If R obtains, each of f, g and l induces a 50:50 chance of obtaining \$10 or \$0. If B obtains, each of them induces again the same probability distribution. All three of f, g and l are then represented by the same AA act. As it fails to distinguish between

[1] What this paper calls the AA framework is the account that Fishburn (1970) and Schmeidler (1989) provide rather than the original framework of Anscombe & Aumann (1963). See Section 6 below.

these three acts, the AA framework forces their indifference. In terms of beliefs, it means that the "diagonal" event $\{(H,R),(T,B)\}$ must necessarily be perceived as likely as H or T and therefore as unambiguous (specifically of probability 1/2). In fact, under classic assumptions, this conclusion extends to situations involving distinct AA acts. For instance, consider f and l' and suppose that the Decision-Maker is risk neutral and in particular indifferent between having a 50:50 chance of obtaining \$10 or \$0 and having a 50:50 chance of obtaining \$9 or \$1. This time, the AA framework does see the difference between the two acts as f and l' induce different distributions on the outcomes at each of R and B. Yet, all these distributions being indifferent, the standard AA Monotonicity axiom, which applies the logic of statewise dominance with respect to the ambiguous source, forces f and l' to be indifferent. In other words, l' must essentially be perceived as unambiguous.

In this paper, we challenge such conclusions that diagonal acts like l or l', or diagonal events like $\{(H,R),(T,B)\}$, are perceived as unambiguous. In this respect, what appears to us to be inappropriate in the AA framework is the statewise logic, as applied to the ambiguous source of uncertainty, that is inherent to both the definition of AA acts and the AA Monotonicity axiom. This statewise logic certainly makes sense if it is assumed that the ambiguous source resolves prior to the randomization device. In this case, one may reasonably expect agents to ask themselves how well they would feel at each state in the ambiguous source and apply the corresponding monotonicity condition. But, in our view, this implicit timing of resolution of uncertainty proves in the first place to be too restrictive for ambiguity aversion. Indeed, suppose now that the timing of resolution of uncertainty is reversed so that the randomization device resolves first. Then, at each of H or T, all three of h, k and l yield \$10 or \$0 with the same, yet unknown, probabilities. Monotoncity with respect to the randomization device leads to the conclusion that all three of h, k and l must be pairwise indifferent. Hence, l now appears to be just as ambiguous as h or k. Alternatively, the diagonal event must be perceived as just as ambiguous as events R or B. Finally, simply reversing the timing of resolution yields ambiguity averse choice patterns that are impossible to obtain under the "direct" timing of AA.

To intuitively understand why the timing of resolution of uncertainty interferes with the modeling of ambiguity aversion, note as a preliminary remark that the randomization device is supposed in the first place to allow the agent to randomize ambiguous acts and is modeled with a second state space. It is then also possible for the agent to randomize unambiguous act on events from the ambiguous source of uncertainty. Therefore, ambiguity aversion might in general reveal itself through at least one of the following two choice patterns:

(1) a preference for randomizing two indifferent ambiguous acts on unambiguous events,

(2) an aversion to randomizing two indifferent unambiguous acts on ambiguous events.

Choice Pattern (1) has often been taken to be the defining feature of ambiguity aversion. For instance, see Schmeidler (1989) or Gilboa & Schmeidler (1989). Indeed, the use of an unambiguous event in the randomization between two indifferent ambiguous acts reduces the exposure to ambiguity. It is then natural to expect an ambiguity averse Decision-Maker to have a preference for the resulting randomization. Meanwhile, Choice Pattern (2) appears to be just as plausible as Choice Pattern (1). This time, the use of an ambiguous event in the randomization between two indifferent unambiguous acts increases the exposure to ambiguity. Ambiguity averse Decision-Makers can be expected to disprefer the resulting randomization. In the example, l can be seen as the randomization between h and k on H and T, and Choice Pattern (1) would support a preference for l over each of h and k. Meanwhile, l can also be seen as the randomization between f and g on R and B, and Choice Pattern (2) would support this time a preference for each of f and g over l. Now, the AA direct timing of resolution of uncertainty and their notion of monotonicity make a blind commitment to Choice Pattern (1) and fail to acknowledge the relevance of Choice Pattern (2). The resulting randomization in Choice Pattern (2) is indeed by construction indifferent to each of the initial acts at each state of the ambiguous source and, therefore, by AA monotonicity, indifferent to each of them. However, the experimental literature which we selectively review Section 6 provides some support to Choice Pattern (2).

In this context, the objective of this paper is to build an axiomatic theory of ambiguity aversion that acknowledges the role of Choice Pattern (2) and, in particular, accommodates the ambiguous nature of l in the example. We achieve this by adopting the reversed timing of resolution of uncertainty according to which the randomization device resolves prior to the ambiguous source. To do so, we replace the AA assumption of exogenous probabilities with a state-space description of the randomization device and obtain a theory of ambiguity that is entirely dual to that of Schmeidler (1989): our Decision-Maker is characterized by a capacity on the ambiguous source and a (subjective) probability on the randomization device. What differs from Schmeidler (1989) is only the order of integration in the evaluation of acts. Moreover, and as expected, ambiguity aversion turns out to be equivalent to Choice Pattern (2), in a sense we later explain.

Broadly speaking, which timing is appropriate ultimately depends on the economic application that one has in mind. Both are equally plausible, and it is difficult to think of general arguments in favor of either one. Note also that many decision problems or experimental setups of interest involving multiple sources of uncertainty simply do not provide the information as to which timing is the correct one. In this case, the choice of a timing becomes a purely subjective matter, which makes again the two equally plausible. However, there are arguably methodological reasons to favor the reversed timing when it comes specifically to building a theory of ambiguity aversion. First, our introductory example already suggests that the AA timing imposes unjustified bounds on ambiguity aversion. Our general results confirm this by showing that the reversed timing amplifies ambiguity aversion and leads, for instance, to larger no-trades intervals à la Dow & Werlang (1992). Intuitively, this is because our agent meets ambiguity multiple times (specifically at each of the possible states of the randomization device) while the agent of Schmeidler's agent only meets ambiguity once prior to the resolution of the randomization device. Second, the issue of the timing of resolution of uncertainty is currently receiving increasing attention in the experimental literature. Indeed, early arguments due to Oechssler & Roomets (2014), Bade (2015) and Kuzmics (2017) call the relevance of random incentive mechanisms for the elicitation of ambiguous beliefs into question. In such mechanisms, a subject is given the opportunity to make a decision in each of several decision problems but is only effectively paid according to one of his decisions determined at random by appealing to a randomization device. These authors worry about the possibility that subjects use the randomization device to construct hedges to reduce their exposure to ambiguity and hence fail to report their true ambiguity-averse preferences. But the plausibility of the argument crucially depends on the timing of resolution of uncertainty. As we will see in greater detail, random incentive mechanisms are always compatible with our notion of ambiguity aversion under the reversed timing of resolution of uncertainty.

The remainder of the paper is organized as follows: Section 2 presents the framework and notation we use all along. Section 3 introduces the Expected Choquet Utility (ECU) representation of preferences, which is how we dub our dual version of the Schmeidler (1989) Choquet Expected Utility (CEU) representation. In this section, we compare in detail ECU to CEU but also to the dual model of Bommier (2017). Sections 4 and 5 respectively provide axiomatic characterizations for the ECU representation and various forms of ambiguity aversion within the model. Finally, Section 6 discusses our modeling approach and the related literature. All proofs are gathered in the Appendix.

2 Framework and notation

Consider a state space $\mathcal{S}_1$ representing a source of uncertainty that the Decision-Maker (DM) may perceive as ambiguous. Consider also a second state space $\mathcal{S}_2$ representing the uncertainty attached to some randomization device. We suppose that $\mathcal{S}_2$ is unambiguous. Yet we do not take the probability on $\mathcal{S}_2$ as exogenously given and will rather elicit it from preference. Consider finally a set $\mathcal{X}$ of consequences.

Let $\mathcal{S}$ denote the Cartesian product of $\mathcal{S}_1$ and $\mathcal{S}_2$ and $\mathcal{B}$ denote the Boolean algebra generated by the rectangles $E_1 \times E_2$ for $E_1 \subseteq \mathcal{S}_1$ and $E_2 \subseteq \mathcal{S}_2$. An act is a finitely-valued $\mathcal{B}$-measurable function from $\mathcal{S}$ to $\mathcal{X}$. Let $\mathcal{F}$ denote the set of all acts. More explicitly, a function f from $\mathcal{S}$ to $\mathcal{X}$ lies in $\mathcal{F}$ if there exist two finite partitions Π_1 and Π_2 of $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively such that f is constant on $E_1 \times E_2$ for all $E_1 \in \Pi_1$ and $E_2 \in \Pi_2$. Equivalently, a finitely-valued function f from $\mathcal{S}$ to $\mathcal{X}$ is an act if $f^{-1}(\{x\})$ is a finite disjoint union of rectangles for all $x \in \mathcal{X}$. The DM is endowed with a binary relation $\succsim$ applying to $\mathcal{F}$ representing his preferences on acts.

Our domain $\mathcal{F}$ is the natural adaptation of the domain that Schmeidler (1989) uses in his account of the AA theorem to our setup where the two sources of uncertainty are described in terms of state spaces. Yet this domain may appear to be excessively restricted as, for instance, it typically does not contain the indicator function of a circle. In fact, in our main result, it would be possible to accommodate arbitrary functions from $\mathcal{S}$ to $\mathcal{X}$ at the cost of stronger axioms like adequate versions of Savage's (1954) P7. In contrast, the restriction by $\mathcal{B}$-measurability plays a crucial role in our other results because of its importance for Fubini-type results for finitely additive probabilities or capacities. See Ghirardato (1997) and Marinacci (1997).

For $i = 1, 2$, let $\mathcal{F}_i$ denote the set of all finitely-valued functions from $\mathcal{S}_i$ to $\mathcal{X}$. Let π_i denote the projection of $\mathcal{S}$ on $\mathcal{S}_i$ defined by $\pi_i(s) = s_i$ for all $s \in \mathcal{S}$. We define a binary relation $\succsim_i$ on $\mathcal{F}_i$ by setting, for all $f_i, g_i \in \mathcal{F}_i$,

$$f_i \succsim_i g_i \iff f_i \circ \pi_i \succsim g_i \circ \pi_i.$$

Hence, $\succsim_i$ represents the marginal preferences of the DM relative to the single source $\mathcal{S}_i$ of uncertainty.

Following the usual abuse of notation, we will identify every outcome $x \in \mathcal{X}$ with the acts in $\mathcal{F}$, $\mathcal{F}_1$ and $\mathcal{F}_2$ constantly equal to x. Likewise, for $i = 1, 2$, we will identify each $f_i \in \mathcal{F}_i$ with the act $f_i \circ \pi_i \in \mathcal{F}$ and hence treat $\mathcal{F}_1$ and $\mathcal{F}_2$ as subsets of $\mathcal{F}$. We will also identify each $E_i \subseteq \mathcal{S}_i$ with the subset $E_i \times \mathcal{S}_{-i} \in \mathcal{B}$ and hence treat $\mathcal{S}_1$ and $\mathcal{S}_2$ as elements of $\mathcal{B}$.

Considering any two acts $f, g \in \mathcal{F}$ and event $E \in \mathcal{B}$, $f_E g$ denotes the element of $\mathcal{F}$ which is equal to f on E and equal to g on $\mathcal{S} \setminus E$. For instance, and by the previous paragraph, for $f_1, g_1 \in \mathcal{F}_1$ and $E_2 \subseteq \mathcal{S}_2$, $f_{1E_2} g_1$ represents the more cumbersome $f_1 \circ \pi_{1\mathcal{S}_1 \times E_2} g_1 \circ \pi_1$.

A binary relation $\succsim_\ell$ on $\mathcal{B}$ is defined in the usual way: For all $E, F \in \mathcal{B}$,

$$E \succsim_\ell F \iff (\text{there exist } x, y \in \mathcal{X} \text{ such that } x \succ y \text{ and } x_E y \succsim x_F y).$$

A ranking $E \succsim_\ell F$ means that the DM considers event E to be at least as likely as event F, and the relation $\succsim_\ell$ is referred to as the DM's comparative likelihood ranking of events.

3 Expected Choquet Utility

We now present the Expected Choquet Utility representation of preferences that the next section characterizes axiomatically. Special emphasis is put on the connections to the Schmeidler (1989) model of Choquet Expected Utility (CEU) preferences and the Bommier (2017) dual approach to CEU. We start with preliminary definitions.

Consider a set $\mathcal{E}$ endowed with a Boolean algebra $\mathcal{B}_\mathcal{E}$ of subsets. A *probability measure* on $(\mathcal{E}, \mathcal{B}_\mathcal{E})$ is a function P from $\mathcal{B}_\mathcal{E}$ to $[0, 1]$ such that $P(\mathcal{E}) = 1$ and $P(E \cup F) = P_2(E) + P_2(F)$ for all $E, F \in \mathcal{B}_\mathcal{E}$ such that $E \cap F = \varnothing$. Moreover, a probability measure P on $(\mathcal{E}, \mathcal{B}_\mathcal{E})$ is said to be *convex-ranged* if for all $E \in \mathcal{B}_\mathcal{E}$ satisfying $P(E) > 0$ and all $\alpha \in (0, P(E))$, there exists $F \subseteq E$ such that $P(F) = \alpha$.

A *capacity* on $(\mathcal{E}, \mathcal{B}_\mathcal{E})$ is a function v from $\mathcal{B}_\mathcal{E}$ to $[0, 1]$ such that $v(\mathcal{E}) = 1$, $v(\varnothing) = 0$, and $v(E) \geq v(F)$ for all $E, F \in \mathcal{B}_\mathcal{E}$ such that $F \subseteq E$. If v is a capacity on $(\mathcal{E}, \mathcal{B}_\mathcal{E})$ and ζ is a bounded function from $\mathcal{E}$ to $\mathbb{R}$ such that $\{\zeta \geq t\} \in \mathcal{B}_\mathcal{E}$ for all $t \in \mathbb{R}$, the *Choquet integral* of ζ with respect to v is defined in the following way:

$$\int_\mathcal{E} \zeta(s)\, dv(s) = \int_{-\infty}^{0} [v(\zeta \geq t) - 1]\, dt + \int_0^{+\infty} v(\zeta \geq t)\, dt.$$

Moreover, we say that a capacity (or a probability) is defined on $\mathcal{E}$ instead of $(\mathcal{E}, \mathcal{B}_\mathcal{E})$ when $\mathcal{B}_\mathcal{E} = 2^\mathcal{E}$.

Suppose v_1 is a capacity on $\mathcal{S}_1$, P_2 is a probability measure on $\mathcal{S}_2$, and u is a function from $\mathcal{X}$ to $\mathbb{R}$. Then, we say that (v_1, P_2, u) provides an *Expected Choquet Utility* (ECU) representation of $\succsim$ if, for all $f, g \in \mathcal{F}$,

$$f \succsim g \iff \int_{\mathcal{S}_2} \int_{\mathcal{S}_1} u \circ f(s_1, s_2)\, dv_1(s_1) dP_2(s_2) \geq \int_{\mathcal{S}_2} \int_{\mathcal{S}_1} u \circ g(s_1, s_2)\, dv_1(s_1) dP_2(s_2).$$

The DM portrayed in an ECU representation faces a source of uncertainty that is represented by $\mathcal{S}_1$ and that he may perceive as ambiguous. He also disposes of a randomization device that is represented by $\mathcal{S}_2$ and that may be used for randomizing the ambiguous acts defined over $\mathcal{S}_1$. Furthermore, he considers that the randomization device resolves prior to the ambiguous source of uncertainty. For instance, this may be because he has the information that the randomization device truly resolves first or because he simply happens to believe that it does so. Another possibility is that he uses this timing, not for its descriptive accuracy, but rather in a more pragmatic way, because it allows greater levels of ambiguity aversion than those permitted by the assumption that the ambiguous source resolves first, as explained in greater detail in the next section. In any case, the DM uses this specific timing directly in the evaluation of acts as depicted in the ECU representation, and more specifically in the fact that the outer integral is on $\mathcal{S}_2$, while the inner integral is on $\mathcal{S}_1$. This suggests indeed that he analyses each act $f \in \mathcal{F}$ by asking himself how well he would feel when choosing f and observing each of the possible state of the randomization device $\mathcal{S}_2$ and then aggregating on $\mathcal{S}_2$, which only makes sense under the assumption of this specific timing. The outer integral is a Lebesgue integral with respect to an endogenously determined probability measure P_2, which reveals the unambiguous nature of the randomization device, while the inner integral is with respect to a (possibly non-additive) capacity v_1, which allows the possibility of sensitivity to ambiguity on $\mathcal{S}_1$.

For expositional convenience, we define the ECU functional in the following way: for all $f \in \mathcal{F}$,

$$ECU(f) := \int_{\mathcal{S}_2} \int_{\mathcal{S}_1} u \circ f(s_1, s_2)\, dv_1(s_1) dP_2(s_2).$$

This representation turns out to be a particular case of that of Sarin & Wakker (1992). When adequately reformulated in our more specific setup, their theorem provides a representation where the DM has beliefs in the form of a capacity v on $\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$ with an additive $\mathcal{S}_2$-marginal and evaluates acts by the Choquet integral on $\mathcal{S}$ with respect to v. Now, as shown in the proof of Proposition 1 in Appendix B, for all $f \in \mathcal{F}$, we have

$$ECU(f) = \int_{\mathcal{S}} u \circ f(s)\, dv(s),$$

where v denotes the capacity on $(\mathcal{S}, \mathcal{B})$ defined, for all $E \in \mathcal{B}$, by

$$v(E) = \int_{\mathcal{S}_2} \int_{\mathcal{S}_1} \mathbf{1}_E(s_1, s_2)\, dv_1(s_1) dP_2(s_2).$$

The simple possibility of reformulating the ECU representation in this way suggests the following: it is equivalent to consider that the DM uses the assumption that the unambiguous source of uncertainty resolves first, not directly in the evaluation of acts as explained in the previous paragraph, but rather in the evaluation of the likelihood of events as depicted in v, and then applies the logic of Choquet integration with respect to v. As we will see, that the two approaches yield the same evaluations is remarkable and far from obvious.

The ECU representation is dual to the CEU representation of Schmeidler (1989) in a relatively straightforward way. Indeed, though Schmeidler uses the AA framework, his decision criterion can be reformulated in our setup as the maximization of

$$CEU(f) \;:=\; \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} u \circ f(s_1, s_2)\; dP_2(s_2) dv_1(s_1).$$

See Grabisch *et al.* (2022) for an axiomatization of this criterion in our current setup. It is hence clear that the major difference between the Schmeidler (1989) model and our approach lies in the timing of resolution of uncertainty. While he assumes that the randomization device $\mathcal{S}_2$ resolves after the ambiguous source $\mathcal{S}_1$, we consider that it does before.

In a rather surprising way, our approach also turns to be dual to that of Bommier (2017), which is already supposed itself to be dual to that of Schmeidler. To clarify, note first that the Schmeidler (1989) criterion can be reformulated for an act $f \in \mathcal{F}$ such that $u \circ f$ has values in the closed interval $I \subseteq \mathbb{R}$ in the following way:

$$CEU(f) \;=\; \int_{\mathcal{S}_1} \int_I P_2(u \circ f(s_1, \cdot) \geq t) dt dv_1(s_1) \;+\; min(I).$$

Bommier uses the AA framework and obtains a wide class of models that are dual to AA monotonic models. His Choquet model is only a particular case. When adequately reformulated in our setup, Bommier's Choquet case simply consists in changing the order of integration in the previous formula and leads to the following criterion:

$$\overline{CEU}(f) \;:=\; \int_I \int_{\mathcal{S}_1} P_2(u \circ f(s_1, \cdot) \geq t) dv_1(s_1) dt \;+\; min(I).$$

As it turns out, Bommier's approach is also a particular case of Sarin & Wakker (1992). Indeed, as shown in the proof of Proposition 1 in Appendix B, for every $f \in \mathcal{F}$, we have,

$$\overline{CEU}(f) \;=\; \int_{\mathcal{S}} u \circ f(s)\; d\overline{v}(s),$$

where $\overline{v}$ denotes the capacity on $(\mathcal{S}, \mathcal{B})$ defined, for all $E \in \mathcal{B}$, by

$$\overline{v}(E) \;=\; \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} \mathbf{1}_E(s_1, s_2)\; dP_2(s_2) dv_1(s_1).$$

This reformulation reveals a major difference between Schmeidler's and Bommier's approaches. The two use the same timing of resolution of uncertainty, namely that one according to which the ambiguous source resolves first. But Schmeidler's DM uses this timing directly in the evaluation of acts, while Bommier's one uses it in the evaluation of the likelihood of events and applies then the logic of Choquet integration. As already proven by an example of Sarin & Wakker (1992) and made more systematic by Bommier (2017), the two approaches do not coincide. The difference contrasts with what happens under our reversed timing of resolution of uncertainty, where the two approaches agree with each other. This asymmetry is finally key in understanding why ECU is dual to each of CEU and $\overline{CEU}$. First, ECU and CEU are dual to each other in that they rely on the two different timings of resolution of uncertainty specifically in the evaluation of acts. But ECU and $\overline{CEU}$ are also dual to each other in that they rely on the two different timings specifically in the evaluation of the likelihood of events. (See also the proof of Proposition 5 in Appendix B for alternative formulations of CEU, $\overline{CEU}$ and ECU, which make the three criteria directly and globally comparable).

Now, key differences between the three functionals ECU, CEU and $\overline{CEU}$ can be illustrated within our introductory example. Fix $\alpha \in (0, 1/2)$ and suppose $v_1(R) = v_1(B) = \alpha$ and $P_2(H) = P_2(T) = 1/2$. Suppose also $u = Id$. Then, the ECU representation imposes the following preference ranking:

$$f \sim \ g \ \succ \ l \sim \ h \sim \ k.$$

More specifically, the utility value of each of f and g is given by 5, while the utility value of each of h, k and l is given by 10α. Hence, as expected, our representation essentially makes l an ambiguous act. This contrasts with the predictions of each of CEU and $\overline{CEU}$, which both suppose that acts in $\mathcal{F}$ inducing the same AA act must be indifferent and hence lead to the following preference pattern:

$$f \sim \ g \ \sim \ l \succ \ h \sim \ k.$$

This time, for both CEU and $\overline{CEU}$, the utility value of f, g and l is equal to 5, while that of h and k is still given by 10α. In other words, l becomes an unambiguous act.

The next proposition provides a more systematic analysis of these cases where the different functionals ECU, CEU and $\overline{CEU}$ agree with each other. It relies on the following definitions. For $i \in \{1, 2\}$, we say that two acts $f_i, g_i \in \mathcal{F}_i$ are ***comonotonic*** if there are no $s_i, s_i' \in \mathcal{S}_i$ such that $f_i(s_i) \succ f_i(s_i')$ and $g_i(s_i') \succ g_i(s_i)$. This notion is used by Schmeidler (1989) to obtain his axiomatic characterization of CEU. Intuitively, two comonotonic acts vary in the same direction and hence they cannot provide hedges for each the other. An act $f \in \mathcal{F}$ is said to have ***comonotonic $\mathcal{S}_i$-sections*** if $f(s_i, \cdot)$ and $f(s_i', \cdot)$ are comonotonic for all $s_i, s_i' \in \mathcal{S}_i$. Ghirardato (1997) uses these notions to obtain Fubini-type results for Choquet integrals, which is key for the following result. In fact, there is a different but equivalent way to define the property of comonotonic $\mathcal{S}_1$-sections that we may use. For all $f_2, g_2 \in \mathcal{F}_2$, we write $f_2 \trianglelefteq g_2$ if, for all $x \in \mathcal{X}$, we have $\{f_2 \succsim_2 x\} \supseteq \{g_2 \succsim_2 x\}$ or $\{f_2 \succsim_2 x\} \subseteq \{g_2 \succsim_2 x\}$; equivalently, the preference upper sets of f_2 and g_2 form a chain. Then, an act $f \in \mathcal{F}$ has comonotonic $\mathcal{S}_1$-sections if and only if $\trianglelefteq$ is a complete binary relation on the $\mathcal{S}_1$-sections of f. This reformulation turns out to be directly comparable to the property of stochastically ordered $\mathcal{S}_1$-sections that we introduce next. For $f_2, g_2 \in \mathcal{F}_2$, we say that f_2 ***stochastically dominates*** g_2, and write $f_2 \trianglerighteq g_2$, if $\{g_2 \succsim_2 x\} \succsim_\ell \{f_2 \succsim_2 x\}$ for all $x \in \mathcal{X}$. Now, for $f \in \mathcal{F}$, we say that f has ***stochastically ordered*** $\mathcal{S}_1$***-sections*** if the binary relation $\trianglerighteq$ is complete and hence a weak order on the $\mathcal{S}_1$-sections of f. We now have all the ingredients we need to formally identify our special cases of agreement between the three representations CEU, $\overline{CEU}$ and ECU.

Proposition 1 *The following statements hold for all* $f \in \mathcal{F}$*:*

- *If* f *has comonotonic* $\mathcal{S}_i$*-sections for some* $i \in \{1, 2\}$*, then* $ECU(f) = \overline{CEU}(f)$*,*
- *If* f *has comonotonic* $\mathcal{S}_2$*-sections, then* $ECU(f) = CEU(f)$*,*
- *If* f *has stochastically ordered* $\mathcal{S}_1$*-sections, then* $CEU(f) = \overline{CEU}(f)$*.*

All three of the assumptions of comonotonic $\mathcal{S}_1$-sections, comonotonic $\mathcal{S}_2$-sections and stochastically ordered $\mathcal{S}_1$-sections capture the intuition that the sections of the act provide no hedge on either $\mathcal{S}_1$ or $\mathcal{S}_2$, though they do so in different formal ways. Their negations reveal the presence of different forms of hedges. Broadly speaking, Proposition 1 then shows that disagreements between the evaluations of an act reveal the presence of hedges among its sections. Those hedges obtained by negating the comonotonicity of sections are referred to as (*statewise*) *hedges* and those ones obtained by negating the stochastic orderedness as *stochastic hedges*.

Proposition 1 can be illustrated in a large extent within the following refinement of the introductory example. Consider the acts defined as follows:

f'	R	B
H	\$10	\$9
T	\$0	\$1

l'	R	B
H	\$10	\$1
T	\$0	\$9

h'	R	B
H	\$10	\$1
T	\$9	\$0

f''	R	B
H	\$10	\$9
T	\$1	\$0

l''	R	B
H	\$10	\$0
T	\$1	\$9

h''	R	B
H	\$10	\$0
T	\$9	\$1

In the analysis of this example, we suppose that $\alpha < 1/2$. This value makes the capacity on $\{R, B\}$ convex, a feature commonly associated to ambiguity aversion since Schmeidler (1989). The evaluations of the various acts relatively to CEU, $\overline{CEU}$ and ECU are as follows:

	5	$\alpha + 9/2$	$9\alpha + 1/2$
CEU	f', l'	f'', l''	h', h''
$\overline{CEU}$		f', f'', l', l''	h', h''
ECU		f', f''	l', l'', h', h''

Here, f', f'' and h' have comonotonic $\mathcal{S}_1$-sections whereas f'', h' and h'' have comonotonic $\mathcal{S}_2$-sections and f'', l'' , h' and h'' have stochastically ordered $\mathcal{S}_1$-sections.

The first item of Proposition 1 shows that $\overline{CEU}$ and ECU only differ in their treatment of acts whose $\mathcal{S}_1$-sections and $\mathcal{S}_2$-sections both provide a hedge. Such acts are typically obtained by starting from other acts with either comonotonic $\mathcal{S}_1$-sections or comonotonic $\mathcal{S}_1$-sections and then applying permutations within rows or within columns. By the first item of the proposition, $\overline{CEU}$ and ECU agree in their evaluation of these initial acts, but they react very differently to permutations within rows or within columns. Indeed, ECU does not react to a permutation within row while $\overline{CEU}$ does not react to a permutation within column (which leaves the induced AA act unchanged). For instance, $\overline{CEU}$ and ECU agree in their evaluation of f'. Moreover, l' provides a double hedge and is obtained from f' through a permutation within column. As expected, $\overline{CEU}$ is indifferent to the permutation, but ECU reacts negatively. This is because the permutation results in a hedge discounting the high gains that were resulting for sure on H with f'. For another example, $\overline{CEU}$ and ECU agree in their evaluation of h''. In addition, l'' provides a double hedge and is obtained from h'' through a permutation within row. This time, ECU is indifferent to the permutation, but $\overline{CEU}$ reacts positively. This is because the permutation results in a hedge on $\mathcal{S}_1$ that reduces the exposure to ambiguity.

The second item of the proposition shows that any disagreement in the utility values of CEU and ECU reveals the presence of a hedge among the $\mathcal{S}_2$-sections. Such acts are typically obtained by starting from other acts with comonotonic $\mathcal{S}_2$-sections and applying a permutation within row. By the second item of the proposition, CEU and ECU agree in their evaluation of these initial acts, but they react very differently to permutations within row. Indeed, ECU does not react as already explained above, but CEU may react negatively. Indeed, consider f'' which is obtained from f' through such a permutation. The latter cancels the perfect hedge on $\mathcal{S}_1$ initially present in f' and hence increases the exposure to ambiguity.

The third item of the proposition shows that any disagreement in the utility values of CEU and $\overline{CEU}$ is due to a stochastic hedge on $\mathcal{S}_1$. Such a hedge is typically lost through a mean preserving contraction within column. For instance, f features such a hedge while f' does not and is obtained from f through a mean preserving contraction within the second column. Now, CEU does not react to the contraction while $\overline{CEU}$ reacts negatively to the loss of a stochastic hedge on $\mathcal{S}_1$ that it implies.

Finally, and incidentally, observe that, since $\alpha < 1/2$, the columns of the table above are ordered by decreasing order. As we move from CEU to $\overline{CEU}$ and then ECU, acts are "pushed" to the right hand side and assigned lower and lower utility values. This seems to suggest that ECU (resp. $\overline{CEU}$) is more ambiguity averse than $\overline{CEU}$ (resp. CEU). Section 5 clarifies the intuition.

4 Axiomatic characterization

This section provides axiomatic foundations to the ECU representation. Our first axiom is a standard condition for a numerical representation to exist. A binary relation $\succsim$ is said to be *complete* if $f \succsim g$ or $g \succsim f$ for all $f, g \in \mathcal{F}$ and *transitive* if, for all $f, g, h \in \mathcal{F}$, $f \succsim g$ and $g \succsim h$ implies $f \succsim h$.

(A1 - Weak Order) $\succsim$ is complete and transitive.

The Sure-Thing Principle is one of Savage's most controversial axiom as it imposes a form of additivity of beliefs that is incompatible with the Ellsberg choices and hence precludes sensitivity to ambiguity. In order to allow for ambiguity to play a role in the decision process, our next axiom only applies Savage's Sure-Thing Principle on events from the unambiguous source $\mathcal{S}_2$ of uncertainty representing the randomization device.

(A2 - Sure-Thing Principle) For all $f, g, h, k \in \mathcal{F}$ and $E_2 \subseteq \mathcal{S}_2$, $f_{E_2}h \succsim g_{E_2}h$ if and only if $f_{E_2}k \succsim g_{E_2}k$.

Interestingly, A2 is much stronger than the form of the Sure-Thing Principle needed to axiomatize CEU in our current setup. See, for instance, Axiom A2 of Grabisch *et al.* (2022). Our A2 applies indeed to all acts in $\mathcal{F}$ while the axiom of Grabisch *et al.* only applies to acts in $\mathcal{F}_2$. This allows us to obtain, by standard arguments, a collection $\{\succsim_{E_2}, E_2 \subseteq \mathcal{S}_2\}$ of binary relations on $\mathcal{F}$ satisfying the following conditions:

(Consequentialism) For all $f, g \in \mathcal{F}$ and $E_2 \subseteq \mathcal{S}_2$, if $f = g$ on $\mathcal{S}_1 \times E_2$, then, $f \sim_{E_2} g$,

(Dynamic Consistency) For all $f, g \in \mathcal{F}$ and all finite partition Π_2 of $\mathcal{S}_2$, if $f \succsim_{E_2} g$ for all $E_2 \in \Pi_2$, then $f \succsim g$.

The next axiom has two fairly standard parts comparable in spirit to Savage's P3. We introduce it by providing first the adequate formulation of the classic AA Monotonicity axiom in our framework, which requires the following: For all $f, g \in \mathcal{F}$, if $f(s_1, \cdot) \succsim_2 g(s_1, \cdot)$ for all $s_1 \in \mathcal{S}_1$, then $f \succsim g$. Hence, the next axiom is composed of two parts corresponding to two asymmetric versions of such monotonicity: A3(i) is a dual version of AA Monotonicity that applies with respect to the randomization device instead of the ambiguous source of uncertainty while A3(ii) is a version of AA Monotonicity restricted to acts depending only on the ambiguous source $\mathcal{S}_1$.

(A3 - Monotonicity)

(i) For all $f, g \in \mathcal{F}$, if $f(\cdot, s_2) \succsim_1 g(\cdot, s_2)$ for all $s_2 \in \mathcal{S}_2$, then $f \succsim g$.

(ii) For all $f_1, g_1 \in \mathcal{F}_1$, if $f_1(s_1) \succsim g_1(s_1)$ for all $s_1 \in \mathcal{S}_1$, then $f_1 \succsim_1 g_1$.

It is known in the literature that imposing simultaneously the classic AA Monotonicity axiom and its dual version A3(i) precludes sensitivity to ambiguity. For instance, see Ceron & Vergopoulos (2021), Grabisch *et al.* (2022) or Proposition 6 below. This explains why we need to weaken the conjunction of the two. Now, by sticking to A3(i), we maintain the full dual version of AA Monotonicity, which is of critical importance to our whole approach. As suggested by the introduction, this is a key feature of our dual approach to ambiguity. Indeed, A3(i) presupposes the specific timing of resolution of uncertainty according to which the randomization device resolves prior to the ambiguous source.

Moreover, A3(i) implies the following form of stochastic independence between the two sources of uncertainty that invokes the conditional preferences implied by A2 as explained above:

(Stochastic Independence) For $f_1, g_1 \in \mathcal{F}_1$ and all $E_2 \subseteq \mathcal{S}_2$, if $f_1 \succsim_1 g_1$, then $f_1 \succsim_{E_2} g_1$.

In words, observing an event in the randomization device does not change the preference relative to acts depending only on the ambiguous source of uncertainty. In contrast, the characterization of CEU due to Grabisch *et al.* (2022) uses the dual version of Stochastic Independence. See their Proposition 4. In particular, consider $E_1, F_1 \subseteq \mathcal{S}_1$ and $E_2 \subseteq \mathcal{S}_2$. Then, A3(i) implies:

$$E_1 \succsim_\ell F_1 \implies E_1 \times E_2 \succsim_\ell F_1 \times E_2.$$

The intuition is similar: observing an event in $\mathcal{S}_2$ does not affect the DM's *ex ante* comparative likelihood ranking on events from $\mathcal{S}_1$. Note also that A3(i) implies the following form of indifference to indifferent ambiguous acts in $\mathcal{F}_1$ on unambiguous events in $\mathcal{S}_2$: For all $f_1, g_1 \in \mathcal{F}_1$ and $E_2 \subseteq \mathcal{S}_2$ such that $f_1 \sim_1 g_1$, we have $f_{1E_2}g_1 \sim f_1 \sim_1 g_1$. In the language of Section 1, this means that A3(i) dismisses the relevance of Choice Pattern (1).

The next axiom is a version of Savage's axiom P4 of Comparative Probability. Specifically, it is weaker than Savage's P4 in that it only applies to the unambiguous events in $\mathcal{S}_2$. However, it is also stronger in that the prizes obtained on these unambiguous events in $\mathcal{S}_2$ are not forced to be constant acts, but allowed to be arbitrary ambiguous acts in $\mathcal{F}_1$.

(A4 - Comparative Probability) For all $f_1^*, f_1, g_1^*, g_1 \in \mathcal{F}_1$ such that $f_1^* \succ_1 f_1$ and $g_1^* \succ_1 g_1$ and all $E_2, F_2 \subseteq \mathcal{S}_2$, $f^*_{1E_2}f_1 \succsim f^*_{1F_2}f_1$ if and only if $g^*_{1E_2}g_1 \succsim g^*_{1F_2}g_1$.

Allowing for arbitrary prizes in $\mathcal{F}_1$ in this axiom provides the following characterization of the comparative likelihood ranking for events in $\mathcal{S}_2$: For all $E_2, F_2 \subseteq \mathcal{S}_2$,

$$E_2 \succsim_\ell F_2 \;\Leftrightarrow\; (\text{there exist } f_1^*, f_1 \in \mathcal{F}_1 \text{ such that } f_1^* \succ_1 f_1 \text{ and } f_1{}^*_{E_2}f_1 \succsim f_1{}^*_{F_2}f_1).$$

This turns out in particular to imply a form of stochastic independence dual to that already implied by A3(i). Indeed, one can prove that for all $E_1 \subseteq \mathcal{S}_1$ and $E_2, F_2 \subseteq \mathcal{S}_2$, A4 implies (up to the previous axioms) the following property:

$$E_2 \succsim_\ell F_2 \implies E_1 \times E_2 \succsim_\ell E_1 \times F_2.$$

In words, observing E_1 in $\mathcal{S}_1$ does not affect the DM's *ex ante* beliefs on events from $\mathcal{S}_2$. Hence, even though imposing AA Monotonicity and its dual form A3(i) proves to be too restrictive for sensitivity to ambiguity at the level of preferences on acts, the conjunction of A3(i) and A4 provides a symmetric treatment of stochastic independence when it comes more specifically to the comparative likelihood ranking.

Our A4 is similar in spirit to Ergin and Gul's (2009) Axiom 5b which is key for their notion of second-order probabilistic sophistication. They too have preferences on acts defined on a product state space $\Omega_a \times \Omega_b$. Their Axiom 5b adapts, not Savage's P4 itself, but rather Machina and Schmeidler's (1992) stronger version P4*. It does so by restricting P4* to events in Ω_b and allowing arbitrary acts on Ω_a instead of constant acts. Hence, our A4 may appear to be to Savage's P4 exactly what their Axiom 5b is to Machina and Schmeidler's P4*. However, there is also a subtle difference. Indeed, in Ergin and Gul's second-order probabilistic sophistication, Ω_a plays the role of the randomization device and Ω_b plays the role of what we call here the ambiguous source of uncertainty. (This is for instance confirmed by the fact that their induced AA acts are defined on Ω_b.) Hence, in our terminology, Ergin and Gul's Axiom 5b restricts P4* to events from the ambiguous source of uncertainty while A4 restricts P4 to events from the randomization device. Likewise, Axiom 5b extends P4* to acts depending on the randomization device while A4 extends P4 to acts depending on the ambiguous source of uncertainty. In this respect, Axiom 5b and A4 appear to implement logics that are dual to each other.

Our fifth axiom corresponds to the usual requirement of non-triviality.

(A5 - Non-triviality) There exist $x, y \in \mathcal{X}$ such that $x \succ y$.

Our next axiom is a version of Savage's axiom P6 of Small Event Continuity. Savage's axiom essentially requires the state space to be infinitely rich: there must exist arbitrarily fine partitions of the state space. In the AA framework too, such a requirement is implicit in the assumption that all probability distributions of outcomes are feasible. Contrary to AA, we model explicitly the randomization device and hence need to invoke explicitly A6 to obtain the inifinite richness of the randomization device in the previous sense. This will allow us in exchange, as in AA, to accommodate arbitrary (finite or infinite) ambiguous sources of uncertainty.

(A6 - Small Event Continuity) For all $f, g, h \in \mathcal{F}$ such that $f \succ g$, there exists a finite partition Π_2 of $\mathcal{S}_2$ such that $f \succ h_{E_2} g$ and $h_{E_2} f \succ g$ for all $E_2 \in \Pi_2$.

Our final axiom is a weak version of Sarin and Wakker's (1992) Cumulative Dominance. For $i = 1, 2$, $f_i \in \mathcal{F}_i$ and $x \in \mathcal{X}$, let $\{f_i \succsim_i x\}$ denote the subset $\{s_i \in \mathcal{S}_i,\ f_i(s_i) \succsim_i x\}$ of $\mathcal{S}_i$. This event collects all states where f_i yields an outcome at least as good as x and is referred to as the cumulative event of f_i at x. Consider two acts, the one defined on $\mathcal{S}_1$ and hence ambiguous, and the other one defined on $\mathcal{S}_2$ and hence unambiguous. Suppose that the DM believes that the two acts have indifferent cumulative events at every outcome. Then, A7 requires the two acts to be indifferent to each other. Put differently, the specific source of uncertainty used in the construction of a cumulative distribution of outcomes by some act is irrelevant for the evaluation of that act.

(A7 - Source Independence) For all $f_1 \in \mathcal{F}_1$ and $f_2 \in \mathcal{F}_2$, if $\{f_1 \succsim_1 x\} \sim_\ell \{f_2 \succsim_2 x\}$ for all $x \in \mathcal{X}$, then $f_1 \sim f_2$.

We now come to our main result:

Theorem 2 *A binary relation $\succsim$ on $\mathcal{F}$ satisfies Axioms A1-A7 if and only if there exist a capacity v_1 on $\mathcal{S}_1$, a convex-ranged probability measure P_2 on $\mathcal{S}_2$, and a non-constant function u from $\mathcal{X}$ to $\mathbb{R}$ such that (v_1, P_2, u) provides an ECU representation of $\succsim$. Moreover, v_1 and P_2 are unique, and u is unique up to positive affine transformation.*

We briefly sketch the proof of Theorem 2, which helps better identify the similarities and differences to Sarin & Wakker (1992). As explained in the previous section, our representation is a particular case of theirs. Yet it is not simply obtained by enriching the Sarin & Wakker (1992) axioms with additional requirements. Indeed, our representation can also be seen as a particular case of that of Savage (1954); That is, for each act $f \in \mathcal{F}$, we have

$$ECU(f) \ = \ \int_{\mathcal{S}_2} U_1[f(\cdot, s_2)] dP_2(s_2),$$

where additionally U_1 is the function from $\mathcal{F}_1$ to $\mathbb{R}$ defined, for all $f_1 \in \mathcal{F}_1$, by

$$U_1(f_1) \ = \ \int_{\mathcal{S}_1} u \circ f_1(s_1) \ dv_1(s_1).$$

Hence, the ECU representation can be obtained by invoking the Savage theorem on the domain $\mathcal{F}_1^{\mathcal{S}_2}$ of functions from $\mathcal{S}_2$ to $\mathcal{F}_1$ while Sarin & Wakker (1992) invoke it on the much smaller domain of unambiguous acts $\mathcal{F}_2$. Practically, this forces us to appeal to stronger axioms: our A3(ii) and our A4. As already explained, these two axioms are the ones forcing a specific treatment of stochastic independence, and are therefore very specific to our approach and timing of resolution of uncertainty. Yet our application of Savage's theorem on a larger domain provides much more and simplifies the rest of the proof: we only have to construct a capacity on $\mathcal{S}_1$ while Sarin and Wakker have to construct one on all of $\mathcal{S}$. This explains why our A3(i) is weaker than a simple restriction of their P3 to ambiguous events and our A7 is weaker than their A4'. The idea here is that we only have to deal with acts in $\mathcal{F}_1$ to obtain v_1, while they have to deal with all of $\mathcal{F}$ to obtain a capacity on $\mathcal{S}$.

5 Ambiguity aversion

The main objective of this section is to define and characterize the notion of ambiguity aversion within the ECU representation as axiomatically characterized in Theorem 2. All along the section, we suppose that $\succsim$ satisfies A1-A7 and denote by v_1, P_2 and u the capacity, probability and utility obtained by applying Theorem 2.

Let CEU, $\overline{CEU}$ and ECU denote the real-valued functionals introduced in the Section 3. For all real-valued functional V on $\mathcal{F}$, $\succsim^V$ denotes the relation induced by V on $\mathcal{F}$ (i.e. $f \succsim^V g \Leftrightarrow V(f) \geq V(g)$ for all $f, g \in \mathcal{F}$). As a consequence, $\succsim^{ECU}$ is just an other way of writing $\succsim$.

Let $\mathcal{N}$ denote the collection of all functionals from $\mathcal{F}$ to $\mathbb{R}$ of the form $f \mapsto \mathbb{E}_P[u \circ f]$ for some probability measure P on $(\mathcal{S}, \mathcal{B})$ of marginal P_2 on $\mathcal{S}_2$. We define $\mathcal{V} := \mathcal{N} \cup \{ECU, CEU, \overline{CEU}\}$. Hence, $\mathcal{V}$ collects the representing functionals for the preferences of agents that either (1) have an ECU representation with respect to (v_1, P_2, u), or (2) have a CEU representation with respect to the same parameters (v_1, P_2, u), or (3) have a $\overline{CEU}$ representation still with respect to these parameters, or finally (4) have a Subjective Expected Utility representation with respect to u and some probability measure on $(\mathcal{S}, \mathcal{B})$

whose $\mathcal{S}_2$-marginal is specifically given by P_2. Note that all such agents have the same marginal preferences on $\mathcal{F}_2$ and hence the same beliefs and risk attitudes toward the (unambiguous) randomization device. As a consequence, these agents possibly differ in terms of their ambiguity attitudes relative to the ambiguous source $\mathcal{S}_1$ of uncertainty.

We now come to the core matter of this section, namely the comparative and absolute definitions of ambiguity aversion. For all $V, W \in \mathcal{V}$, we say that $\succsim^V$ is *more ambiguity averse* than $\succsim^W$ if, for all $f_2 \in \mathcal{F}_2$ and all $f \in \mathcal{F}$,

$$f_2 \succsim^W f \;\Rightarrow\; f_2 \succsim^V f \qquad \text{and} \qquad f_2 \succ^W f \;\Rightarrow\; f_2 \succ^V f.$$

For all $V \in \mathcal{V}$, we say that $\succsim^V$ is *ambiguity averse (loving)* if there exists $W \in \mathcal{N}$ such that $\succsim^V$ is more ambiguity averse than $\succsim^W$ ($\succsim^W$ is more ambiguity averse than $\succsim^V$). A preference which is both ambiguity averse and ambiguity loving is called *ambiguity neutral*.

A preference ranking $f_2 \succsim^W f$ (or $f_2 \succ^W f$) means that Agent W refuses to expose himself to the ambiguity attached to f and prefers the unambiguous act f_2. Hence, the definition says that the more ambiguity averse agent is one who refuses to expose himself to ambiguity whenever the less ambiguity averse agent does. Next, we hold the view that Subjective Expected Utility represents a case of neutrality to ambiguity. This leads to us call an agent ambiguity averse whenever he is more ambiguity averse than some Subjective Expected Utility agent. These definitions are similar in spirit to those of Ghirardato & Marinacci (2002) but they contrast with those of Epstein (1999) and Epstein & Zhang (2001) which rather take probabilistic sophistication to mean ambiguity neutrality.

The following axiom captures the behavioral content of the assumption of ambiguity aversion under the reversed timing of resolution of uncertainty that we assume:

(A8 - Ambiguity Aversion) For all $f_2 \in \mathcal{F}_2$ and $g \in \mathcal{F}$, if $\{f_2 = x\} \sim_\ell \{g(s_1, \cdot) = x\}$ for all $s_1 \in \mathcal{S}_1$ and $x \in \mathcal{X}$, then $f_2 \succsim g$.

This axiom can be seen as an adequate formalization of what the introduction has called Choice Pattern (2). Indeed, all the acts $g(s_1, \cdot)$ for $s_1 \in \mathcal{S}_1$ are elements of $\mathcal{F}_2$ and hence unambiguous by definition. Moreover, in the antecedent of A8, all these acts are assumed to induce the same lottery and therefore be indifferent to each other. Hence, g can be understood as a mixture of indifferent and unambiguous acts on the various states, or events, of the ambiguous source $\mathcal{S}_1$ of uncertainty. And yet A8 imposes a preference for the unambiguous act f_2 over the possibly ambiguous one g. In other words, A8 simply expresses an aversion to randomizing indifferent unambiguous acts on ambiguous events.

To better understand A8 and how it comes into contradiction with the AA framework, let $L_{f(s_1,\cdot)}$ denote the *lottery induced* by f over $\mathcal{X}$ at s_1 for all $f \in \mathcal{F}$ and $s_1 \in \mathcal{S}_1$; that is $L_{f(s_1,\cdot)}(x) = P_2(f(s_1, \cdot) = x)$ for all $x \in \mathcal{X}$. The functional defined on $\mathcal{S}_1$ by $s_1 \mapsto L_{f(s_1,\cdot)}$ then corresponds to what we shall call the *AA act induced* by f. Then, A8 reformulates as follows: For all $f_2 \in \mathcal{F}_2$ and $g \in \mathcal{F}$, if f_2 and g induce the same AA act, then, $f_2 \succsim g$. It is clear from here that the AA framework is typically incompatible with Axiom A8 as it fails to distinguish between f_2 and g. However, there is a major difference between these two acts; namely, the outcome induced by f_2 is independent of the state in $\mathcal{S}_1$ and hence unambiguous while that induced by g depends in general on the state in $\mathcal{S}_1$. This dependence is certainly a potential source of ambiguity in the evaluation of g and therefore could be sufficient for an ambiguity averse agent to express a preference for f_2 over g, as required by A8.

To illustrate A8, recall our introductory example. As f only depends on $\mathcal{S}_2$, it can be identified with an act in $\mathcal{F}_2$. Meanwhile, and by assumption, we have

$$\begin{array}{lllll} L_{f(R,\cdot)}(\$10) = L_{l(R,\cdot)}(\$10) = 1/2, & L_{f(R,\cdot)}(\$0) = L_{l(R,\cdot)}(\$0) = 1/2, \\ L_{f(B,\cdot)}(\$10) = L_{l(B,\cdot)}(\$10) = 1/2, & L_{f(B,\cdot)}(\$0) = L_{l(B,\cdot)}(\$0) = 1/2. \end{array} \quad (1)$$

In this context, A8 applies and leads to the conclusion that $f \succsim l$. In other words, A8 allows act l to be perceived as ambiguous.

Consider now the following stronger version of A8:

(A8* - Strong Ambiguity Aversion) For all $f_2 \in \mathcal{F}_2$ and $g \in \mathcal{F}$, if $f_2 \sim_2 g(s_1, \cdot)$ for all $s_1 \in \mathcal{S}_1$, then $f_2 \succsim g$.

This axiom is obtained from A8 by replacing the condition of equality between the AA acts induced by f_2 and g by the weaker condition of statewise indifference[2] on $\mathcal{S}_1$. As for A8, A8* can be seen as a formalization of Choice Pattern (2), one expressing an aversion to randomizing unambiguous acts on ambiguous events. Yet A8* has a greater domain of applicability than A8. Indeed, Formula (1) from the previous example does not hold if l is replaced with l'. For instance, we have $L_{l'(B,\cdot)}(\$0) = 0$ while $L_{f(B,\cdot)}(\$0) = 1/2$. Therefore, f and l' cannot be compared using A8. However, l' is clearly ambiguous whereas f is not and one could naturally expect an ambiguity averse agent to express the preference $f \succsim l'$. This ranking can actually be derived from A8* since the two acts satisfy:

$$f(R, \cdot) \sim_2 l'(R, \cdot) \quad \text{and} \quad f(B, \cdot) \sim_2 l'(B, \cdot). \quad (2)$$

The ***core*** of the capacity v_1, denoted by $Core(v_1)$, is defined as the set of all probability measures P_1 on $\mathcal{S}_1$ satisfying $P_1(E_1) \geq v_1(E_1)$ for all $E_1 \subseteq \mathcal{S}_1$.

Proposition 3 *The following statements are equivalent:*

- $\succsim$ *satisfies* A8 (*resp.* A8*),
- $Core(v_1) \neq \emptyset$,
- $\succsim$ *is ambiguity averse.*

[2] Our results remain true if one considers the stronger version of A8* obtained by replacing the condition $f_2 \sim_2 g(s_1, \cdot)$ for all $s_1 \in \mathcal{S}_1$ by $f_2 \succsim_2 g(s_1, \cdot)$ for all $s_1 \in \mathcal{S}_1$. A similar remark holds for axioms A9* and A10* presented hereafter. In the latter case, this change makes A10* the dual of A3(i). We nonetheless prefer the current formulations of the axioms because they highlight the basic tension between AA monotonicity and ambiguity aversion in a straightforward way.

Proposition 3 shows, as expected, the equivalence between ambiguity aversion and each of A8 and A8*. This justifies retrospectively the terminology of "Ambiguity Aversion" used to designate the axiom. The proposition also establishes the equivalence to the non-emptiness of the core of v_1. That latter result is similar to that Ghirardato & Marinacci (2002) who already established the equivalence between ambiguity aversion and the non-emptiness of the $Core(v_1)$ in the case of $\succsim^{CEU}$.

As a corollary of Proposition 3, we obtain the following result showing that all three representations CEU, $\overline{CEU}$ and ECU reveal ambiguity aversion in exactly the same circumstances. As we will momentarily see, what differs in each of the representations is rather the amount of ambiguity aversion that is revealed.

Corollary 4 *The following statements are equivalent:*

- $\succsim$ *is ambiguity averse,*
- $\succsim^{\overline{CEU}}$ *is ambiguity averse,*
- $\succsim^{CEU}$ *is ambiguity averse.*

We are now interested in characterizing the *convexity* of the capacity in the ECU representation. A capacity v_1 on $\mathcal{S}_1$ is said to be convex if $v_1(E_1 \cup F_1) + v_1(E_1 \cap F_1) \geq v_1(E_1) + v_1(F_1)$ for all $E_1, F_1 \subseteq \mathcal{S}_1$. This property is an important feature for different reasons. First, it characterizes specific definitions of ambiguity aversion that have been employed in the literature. See, for instance, Schmeidler (1989). Hence, it plays a significant role in numerous economic applications. See for instance Dow & Werlang (1992). At a different level, Schmeidler (1989) shows that, under convexity, the Choquet integral can be reinterpreted as a minimal expectation with respect to the core in the spirit of the maxmin model of Gilboa & Schmeidler (1989). In this case, ECU can be seen as the integral of such a minimal expectation and can therefore be thought of as a dual version of maxmin.

An act $f \in \mathcal{F}$ is said to be *slice-comonotonic* if it has both comonotonic $\mathcal{S}_1$-sections and comonotonic $\mathcal{S}_2$-sections. Consider the following axioms:

(A9(i) - Preference for $\mathcal{S}_1$-comonotonicity) For all $f, g \in \mathcal{F}$ if $\{f(s_1, \cdot) = x\} \sim_\ell \{g(s_1, \cdot) = x\}$ for all $s_1 \in \mathcal{S}_1$ and $x \in \mathcal{X}$ and f has comonotonic $\mathcal{S}_1$-sections, then $f \succsim g$.

(A9(ii) - Preference for $\mathcal{S}_2$-comonotonicity) For all $f, g \in \mathcal{F}$, if $\{f(s_1, \cdot) = x\} \sim_\ell \{g(s_1, \cdot) = x\}$ for all $s_1 \in \mathcal{S}_1$ and $x \in \mathcal{X}$ and f has comonotonic $\mathcal{S}_2$-sections, then $f \succsim g$.

(A9 - Preference for slice-comonotonicity) For all $f, g \in \mathcal{F}$, if $\{f(s_1, \cdot) = x\} \sim_\ell \{g(s_1, \cdot) = x\}$ for all $s_1 \in \mathcal{S}_1$ and $x \in \mathcal{X}$ and f is slice-comonotonic, then $f \succsim g$.

As already explained in the discussion of Proposition 1, the comonotonicity of $\mathcal{S}_1$-sections (respectively $\mathcal{S}_2$-sections) reveals that the uncertainty across different states in $\mathcal{S}_1$ (resp. $\mathcal{S}_2$) does not hedge itself. Moreover, the premises in A9(i) and A9(ii) are similar to that of A8 and impose the equality between the AA acts induced by f and g. Hence, the two axioms can be rephrased as follows: if two acts induce the same AA act and if one offers no hedges (either on $\mathcal{S}_1$ or $\mathcal{S}_2$), then it is weakly preferred to the other one. In other words, A9(i) and A9(ii) embody some aversion toward uncertainty hedges.

To illustrate, consider acts f and l from the introductory example. Clearly, the sections of l are neither comonotonic on $\mathcal{S}_1$ nor on $\mathcal{S}_2$ and, therefore, l provides uncertainty hedges on both $\mathcal{S}_1$ and $\mathcal{S}_2$. Hence, in the AA framework, the agent is always indifferent between l and the unambiguous act f. Meanwhile, for an ECU agent assigning capacities of 0 to each of R and B, l is evaluated as if it resulted in a sure amount of \$0 at each of H and T. In this case, f is strictly preferred to l.

Let us also recall our example from Section 3 and consider acts f' and l'. The former has comonotonic $\mathcal{S}_1$-sections while the latter does not. Moreover, the two induce the same AA act. Hence, A9(i) applies and provides the conclusion $f' \succsim l'$. In contrast, A9(ii) does not apply here as the $\mathcal{S}_2$-sections of f' are not comonotonic. Yet, in other cases, A9(ii) applies while A9(i) does not. For instance, h' and h'' have each comonotonic $\mathcal{S}_2$-sections and induce the same AA act. Hence, A9(ii) provides $h' \sim h''$ while A9 remains silent because none of h' and h'' has comonotonic $\mathcal{S}_1$-sections. Finally, in still other cases, both axioms apply. For instance, both lead to $f'' \succsim l''$ because f'' and l'' induce the same AA act and f'' is slice-comonotonic.

Axiom A9 is similar in spirit to each of A9(i) and A9(ii) as it also embodies an aversion to hedges. But it is also weaker than each of them since it only applies to slice-comonotonic acts. Furthermore, note that all acts in $\mathcal{F}_2$ are slice-comonotonic. For this reason, each of A9(i), A9(ii) and A9 trivially implies A8 and thus ambiguity aversion. Finally, it is possible to strengthen A9 in exactly the same way that A8* strengthens A8; that is, by considering situations where f and g are statewise indifferent on each state in $\mathcal{S}_1$, and not only when they induce the same AA act. Such a strengthening yields the following axiom A9*, which the next proposition will show to be equivalent to A9 under the ECU representation.

(A9* - Strong preference for slice-comonotonicity) For all $f, g \in \mathcal{F}$, if $f(s_1, \cdot) \sim_2 g(s_1, \cdot)$ for all $s_1 \in \mathcal{S}_1$ and f is slice-comonotonic, then $f \succsim g$.

The next proposition shows that each of the four axioms characterizes the convexity of v_1 in the representation of Theorem 2 and hence how they are all equivalent to each other in the ECU representation. It also shows how convexity characterizes those cases where ECU reveals more ambiguity aversion than each of CEU and $\overline{CEU}$.

Proposition 5 *The following statements are equivalent:*

- $\succsim$ *satisfies Axiom* A9(i) (*resp.* A9(ii)*, resp.* A9*, resp.* A9*)*,
- v_1 *is convex,*
- $\succsim$ *is more ambiguity averse than* $\succsim^{\overline{CEU}}$ (*resp.* $\succsim^{CEU}$).

Broadly speaking, Proposition 5 shows that the convexity of v_1 is equivalent in the ECU representation to various forms of aversion to uncertainty hedges. Such aversion is truly a peculiar feature of our model and its reversed timing of resolution of uncertainty. In an AA framework, ambiguity averse agents are typically attracted by hedges because they scatter the good outcomes over the ambiguous states and hence reduce the exposure to ambiguity. However, in doing so, hedges also necessarily scatter the bad outcomes over the unambiguous events. Under our reversed timing of resolution of uncertainty, ambiguity averse agents are rather sensitive to the latter unpleasant view of uncertainty hedges, and non-sensitive to the former one. To them, hedges only reduce the gains they get on unambiguous events, and this finally explains their aversion to them.

Proposition 5 additionally suggests that ambiguity aversion under ECU and its reversed timing of resolution of uncertainty might have a quantitatively more significant impact in economic applications than under CEU or $\overline{CEU}$. We briefly illustrate this possibility in the context of the Dow & Werlang (1992) no trade intervals. Suppose momentarily that the alternatives are monetary assets; that is, suppose $\mathcal{X} \subseteq \mathbb{R}$ and, for simplicity, $u = Id$. Fix a representation $V \in \mathcal{V}$ and an act $f \in \mathcal{F}$. Following Dow & Werlang (1992), we may interpret $V(f)$ and $-V(-f)$ as the expected returns from buying and selling f respectively. The gap between these two values then corresponds to the range of prices where the DM takes no position in the asset. It is called the ***No-trade Interval*** of f relative to V and formally defined as the (possibly empty) interval:

$$NT_V(f) := (V(f), -V(-f)) .$$

Taken together, Proposition 5 and Lemma 16 show, under the assumption of convexity, that for all $f \in \mathcal{F}$

$$NT_{CEU}(f) \;\subseteq\; NT_{\overline{CEU}}(f) \;\subseteq\; NT_{ECU}(f).$$

In other words, ECU predicts larger no trade intervals than each of CEU and $\overline{CEU}$ under the assumption of convexity. In particular, it is possible to have a nontrivial interval $NT_{ECU}(f)$ even when both $NT_{CEU}(f)$ and $NT_{\overline{CEU}}(f)$ are empty. More generally, Lemma 16 formalizes the natural intuition promoted by Dow & Werlang, that more ambiguity aversion implies larger no trade intervals.

Finally, it is easy to obtain from the previous proposition a characterization of additivity for v_1. The latter uses the following axiom which simply requires acts that induce the same AA act to be indifferent.

(A10 - Ambiguity Neutrality) For all $f, g \in \mathcal{F}$, if $\{f(s_1, \cdot) = x\} \sim_\ell \{g(s_1, \cdot) = x\}$ for all $s_1 \in \mathcal{S}_1$ and all $x \in \mathcal{X}$, then $f \sim g$.

As usual by now, A10 can be strengthened into A10* which applies more generally as soon as the acts f and g are statewise indifferent on each state in $\mathcal{S}_1$. In doing so, A10* proves to be a version of monotonicity dual in spirit to that of A3(i).

(A10* - Dual Monotonicity) For all $f, g \in \mathcal{F}$, if $f(s_1, \cdot) \sim_2 g(s_1, \cdot)$ for all $s_1 \in \mathcal{S}_1$, then $f \sim g$.

Proposition 6 *The following statements are equivalent:*

- $\succsim$ *satisfies Axiom A10* (*resp. A10**),
- $\succsim$ *is ambiguity neutral,*
- v_1 *is additive.*

6 Discussion and related literature

Let $\mathcal{L}(\mathcal{X})$ denote the set of all finitely-supported lotteries on $\mathcal{X}$, $\mathcal{A}_1$ the set of all finitely-valued functions from $\mathcal{S}_1$ to $\mathcal{L}(\mathcal{X})$ and $\mathcal{L}(\mathcal{A}_1)$ the set of all finitely-supported lotteries on $\mathcal{A}_1$. In its original form, the framework of Anscombe & Aumann (1963) deals with preferences applying to $\mathcal{L}(\mathcal{A}_1)$. In this context, the two key axioms that yield an SEU representation are Monotonicity with respect to $\mathcal{S}_1$ and Reversal of Order, a form of indifference between *ex ante* and *ex post* randomizations. In order to accommodate ambiguity aversion in the original AA framework, recent papers as Seo (2009) and Martins da Rocha & Mouallem Rosa (2021) suggest dropping Reversal of Order while sticking to Monotonicity.

Also, Fishburn (1970) and Schmeidler (1989) provide an alternative account of the AA theorem where preferences only apply to $\mathcal{A}_1$. In this context, the two key axioms that yield an SEU representation are Monotonicity (still on $\mathcal{S}_1$) and Independence, a form of separability with respect to the randomization device implicit in $\mathcal{L}(\mathcal{X})$. In his seminal contribution, Schmeidler accommodates ambiguity aversion by weakening Independence but once again sticking to Monotonicity. From there, many subsequent papers followed this approach including Gilboa & Schmeidler (1989) or Ghirardato *et al.* (2004).

Yet several authors challenge such AA monotonic models of ambiguity aversion on various grounds. First, the papers of Eichberger & Kelsey (1996) and Eichberger *et al.* (2016) entail a critique of what we have here called Choice Pattern (1) in the introduction. These authors assume a product structure of the state space $\mathcal{S}_1 \times \mathcal{S}_2$ with objective probabilities on $\mathcal{S}_2$ and identify plausible conditions under which an agent is indifferent to randomizations in the following sense: For all $f_1, g_1 \in \mathcal{F}_1$ and $E_2 \subseteq \mathcal{S}_2$, if $f_1 \sim_1 g_1$, then $f_{1E_2}g_1 \sim f_1 \sim g_1$. This property has already been met in Section 4 and is here implied by A3(i). Now, Eichberger & Kelsey (1996) derive it from their assumption of Device Independence with Symmetric Randomizations (DISR) which assumes the indifference with respect to randomizations between acts in $\mathcal{F}_1$ on events in $\mathcal{S}_2$ of same probability. DISR is hence comparable to our A4. Moreover, Eichberger *et al.* (2016) derive the indifference to randomizations from Dynamic Consistency and Consequentialism, both expressed with respect to $\mathcal{S}_2$, and also a property of stochastic independence between $\mathcal{S}_1$ and $\mathcal{S}_2$. Interestingly, their model also appeals to a third source $\mathcal{S}_3$ and includes the possibility of *ex post* randomizations which occur after the realization of the ambiguous state. Then, ambiguity aversion causes a preference for randomizing ambiguous acts on events from the randomization device used *ex post*. In contrast, our model does not allow for the possibility *ex post* randomizations. Ambiguity aversion rather reveals itself through Choice Pattern (2) and the aversion to randomizing unambiguous acts on ambiguous events.

Second, as already explained in the introduction, Oechssler & Roomets (2014), Bade (2015) and Kuzmics (2017) call the relevance of random incentive mechanisms for the elicitation of ambiguous beliefs into question. See also Baillon *et al.* (2019) and Baillon *et al.* (2022). We now explain how our model and its reversed timing overcome the difficulty. Suppose that a DM needs to make a decision in two problems $C_1^a, C_1^b \subseteq \mathcal{F}_1$. An analyst determines at random the problem according to which the DM is paid. To this effect, a partition $\{E_2^a, E_2^b\}$ of $\mathcal{S}_2$ is used: if E_2^a (resp. E_2^b) obtains, the DM is paid according to his choice in C_1^a (resp. C_1^b). Suppose also that f_1^a and f_1^b are optimal in C_1^a and C_1^b respectively. Then, for all $g_1^a \in C_1^a$ and $g_1^b \in C_1^b$, we have $f_1^a \succsim_1 g_1^a$ and $f_1^b \succsim_1 g_1^b$. The property of Stochastic Independence met in Section 4 implies $f_1^a \succsim_{E_2^a} g_1^a$ and $f_1^b \succsim_{E_2^b} g_1^b$. From there, Consequentialism and Dynamic Consistency (still from Section 4) lead to $f_1^a{}_{E_2^a} f_1^b \succsim g_1^a{}_{E_2^a} g_1^b$. In words, the DM will report his truly preferred options in each of the two decision problems. Note that the argument here can be refined by appealing to a strict version of Dynamic Consistency to show that the DM only reports his truly preferred options.

The lesson here is that our reversed timing of resolution of uncertainty allows us to maintain ambiguity on $\mathcal{S}_1$ and yet have a sufficiently strong version of the Sure-Thing Principle that implies both Consequentialism and Dynamic Consistency with respect to $\mathcal{S}_2$. Then, an adequate notion of Stochastic Independence between the two sources that acknowledges the reversed timing is sufficient to deliver the incentive compatibility of random incentive mechanisms. Models that yield incentive incompatibility fail to have these features. For instance, in Bade (2015), the Sure-Thing Principle is restricted to $\mathcal{F}_2$ à la Grabisch *et al.* (2022). Meanwhile, the notion of stochastic independence used is that of Gilboa & Schmeidler (1989) or Klibanoff (2001). Because $\mathcal{S}_2$ carries a single probability measure, this notion is equivalent to Ceron and Vergopoulos' (2021) product $\otimes_1$ which characterizes dominance with respect to $\mathcal{S}_1$ and hence what we call here AA monotonicity and the direct timing of resolution of uncertainty. Likewise, Kuzmics (2017) commits to the direct timing by simply supposing that the agent has well-defined preferences on AA acts. This is confirmed by his use of Dominance both in his Axiom 1 and in his motivating example. Such Dominance is again what we call here AA monotonicity. Finally, Oechssler & Roomets (2014) use the same notion of stochastic independence as Bade (2015) but acknowledge the possibility of escaping incentive incompatibility by appealing to what they call "superstitious ambiguity attitudes". This corresponds to what Ceron & Vergopoulos (2021) denote $\otimes_2$.

Third, Bommier (2017) also develops a dual approach to ambiguity aversion. As we have seen in Section 3, Bommier's dual approach as specified in the Choquet case is based on the same timing of resolution of uncertainty as that of Schmeidler (1989), but uses it in a different way. In fact, Bommier sticks to the AA framework and weakens AA monotonicity. As a consequence, his approach would still lead to the indifference between f, g and l in our introductory example but would possibly explain a preference for f or g over l'.

Fourth, the experimental literature provides little support to the AA framework, AA Monotonicity and Choice Pattern (1). For instance, Dominiak & Schnedler (2011) report results in the context of our introductory example where approximately half of the ambiguity averse agents surprisingly ignore the hedging opportunity provided by diagonal acts and rather express an indifference between h, k and l. Such indifference is impossible to obtain under the AA timing but is in contrast implied by the reversed timing. In fact, Dominiak and Schnedler define ambiguity aversion as a preference for f and g over h and k. Hence, the indifference between h, k and l that they observe means a preference for f and g over l. Such a preference remains consistent with ambiguity aversion in the sense of Choice Pattern (2). For another example, Oechssler & Roomets (2021) slightly increase the gains delivered by diagonal acts so as to have them dominating each of f and g at each of R and B. Yet approximately one third of their subjects chose f or g against these improved diagonal acts in contradiction with AA monotonicity, thereby revealing again, but in a different and more robust way, the ambiguity they attach to diagonal acts and events. Furthermore, the two treatments of their experiment show that a significant proportion of the individuals choose between h (or k) and l only on the basis of their payments: a slight increase in the payment of each makes it preferred to the other one. This reflects an indifference between the diagonal event and the ambiguous events, a conclusion consistent with ECU. Finally, Oechssler *et al.* (2019) provide their subjects the opportunity to make a choice between h, k and l under the different timings of resolution of uncertainty. Somewhat surprisingly, the different timings proposed do not seem to influence choices. Indeed, under every timing, only one third of their ambiguity averse subjects chose a diagonal act. No matter what timing is announced, the remaining two thirds hence make choices that are consistent with the implications of the reversed timing. Baillon *et al.* (2022) confirm in particular the latter finding.

We now examine possible objections to our approach. On the one hand, it might be objected that it is not necessary to appeal to a randomization device to interpret the AA framework. The latter can simply be understood as a case where the outcome space has a convex structure, like, for instance, that of a convex consumption set. This view would naturally make the issue of the timing of resolution of uncertainty quite irrelevant. However, it is unlikely to get us very far. Think for instance of the AA Independence axiom. What makes it so compelling is the basic intuition that there can be no meaningful complementarity between outcomes obtained on incompatible events. Moscati (2016) explains in detail how this intuition helped Samuelson to accept Independence. Now, defending the AA framework by denying the probabilistic interpretation of the convex structure and appealing to convex consumption sets seems to open the door to meaningful complementarity effects. In short, it will be beer *and* bretzel instead of beer *or* bretzel. It is then no longer clear why an agent would satisfy the AA Independence axiom or its weaker versions like those of Schmeidler (1989) and the subsequent ambiguity literature. Likewise, it is not clear what the AA original axiom of Reversal of Order would mean under this view. Finally, what makes the assumption of a randomization device inevitable is not just the convex structure of the outcome space but also the sound interpretation of the axioms.

On the other hand, our dual theory of ambiguity aversion inevitably suffers from limitations that are dual to those that the Schmeidler (1989) model inherits from its use of the AA framework. For instance, because it sticks to the reversed timing, our approach makes a blind commitment to Choice Pattern (2) and leaves no room for Choice Pattern (1). Yet, it seems clear that a general theory of ambiguity aversion should accommodate both, and this can only be done if no specific timing is assumed. For instance, the one-stage theory of Sarin & Wakker (1992) would allow to implement this alternative project. However, this would run into conceptual difficulties. Indeed, as a key feature, the randomization device and the ambiguous source are stochastically independent from each other; that is, observing the true state in either source does not convey information as to which state will obtain in the other source and hence should not affect the preferences and beliefs relative to this other source. Now, it is known in the literature that imposing the full requirement symmetrically across the two sources is too restrictive and forbids for instance the Ellsberg (1961) choice pattern. See Ceron & Vergopoulos (2021). But it is nonetheless possible to impose an asymmetric version of this requirement. And which asymmetric version is appropriate to impose depends on the timing of resolution: if, for instance, the randomization device resolves first, then it makes sense to suppose that observing the outcome of the randomization device does not affect the preference and beliefs relative to the ambiguous source. Finally, the point here is that the assumption of a specific timing is needed to model the stochastic independence between the two sources. Without it, one-stage approaches to ambiguity leave this important informational feature of the decision problem out of the model and condemn themselves to preferences and beliefs relative to the one source depending on the state in the other source, a hardly compelling motive in our view. An alternative route could consist in including the possibility of each timing by appealing to Saito (2015). However, it is again not clear if this route would deliver an adequate notion of stochastic independence, or remain compatible with the use of random incentive mechanisms.

Finally, we close with a remark of independent interest. In a recent contribution, Hartmann (2020) shows that Savage's P3 is redundant in the sense that it is implied by the other axioms. However, if one restricts preferences to finitely-valued acts, then Savage's P7 is no longer required. In this case, the axioms no longer imply P3 which must therefore still be explicitly postulated. Our proof of Theorem 2 nonetheless shows that "half" of P3 is still redundant and that it is possible to obtain it from the other axioms. Concretely, this explains why our A3(i) only assumes an implication, as opposed to requiring additionnally a strict preference $f \succ g$ whenever $f(\cdot, s_2) \succ g(\cdot, s_2)$ for all s_2 in some nonnull subset.

7 Appendices

Appendix A – Proof of Theorem 2

Elements of $\mathcal{F}$ being finitely-valued we have that $f(\cdot, s_2) \in \mathcal{F}_1$ for all $f \in \mathcal{F}$ and all $s_2 \in \mathcal{S}_2$. Let Φ be the function from $\mathcal{F}$ to $\mathcal{F}_1^{\mathcal{S}_2}$ defined by $\Phi(f)(s_2) = f(\cdot, s_2)$ for all $f \in \mathcal{F}$ and $s_2 \in \mathcal{S}_2$. Φ is clearly injective and thus we can define without ambiguity the image $\succsim^*$ of $\succsim$ on $\Phi(\mathcal{F})$ by setting for all $F, G \in \Phi(\mathcal{F})$,

$$F \succsim^* G \iff \Phi^{-1}(F) \succsim \Phi^{-1}(G).$$

Lemma 7 *If $\succsim$ satisfies Axioms A1, A2, A3(i), A4, A5, A6, then there exists a convex-ranged probability measure P_2 on $\mathcal{S}_2$ and a non-constant function U_1 from $\mathcal{F}_1$ to $\mathbb{R}$ such that for all $f, g \in \mathcal{F}$,*

$$f \succsim g \iff \int_{\mathcal{S}_2} U_1[f(\cdot, s_2)]\ dP_2(s_2) \ \geq \int_{\mathcal{S}_2} U_1[g(\cdot, s_2)]\ dP_2(s_2). \tag{3}$$

Moreover, P_2 is unique, and U_1 is unique up to positive affine transformations.

Proof. Since acts are finitely-valued and $\mathcal{B}$-measurable, $\Phi(\mathcal{F})$ is the set of all the finitely-valued functions from $\mathcal{S}_2$ to $\mathcal{F}_1$. Observe that $\Phi(f_{E_2}g) = \Phi(f)_{E_2}\Phi(g)$ for all $f, g \in \mathcal{F}$ and $E_2 \subseteq \mathcal{S}_2$. Axioms A1, A2, A4 and A6 then imply that $\succsim^*$ satisfies P1, P2, P4 and P6 of Savage (1954). In parallel, note that A5 obviously implies the non-triviality of $\succsim^*$. Therefore, it is sufficient to prove that A3(i) implies that $\succsim^*$ satisfies Savage's P3 to invoke the Savage Theorem.

Let us say that an event $E_2 \subseteq \mathcal{S}_2$ is ***null*** if $f_{E_2}h \sim g_{E_2}h$ for all $f, g, h \in \mathcal{F}$. Savage's P3 applied to $\succsim^*$ then reformulates as follows:

(A3*(i)) For all non-null $E_2 \subseteq \mathcal{S}_2$, all $f_1, g_1 \in \mathcal{F}_1$ and all $h \in \mathcal{F}$, $f_1 \succsim_1 g_1$ if and only if ${f_1}_{E_2}h \succsim {g_1}_{E_2}h$.

Fix a non-null $E_2 \subseteq \mathcal{S}_2$, $f_1, g_1 \in \mathcal{F}_1$ and $h \in \mathcal{F}$. By A3(i), if $f_1 \succsim_1 g_1$, then, we immediately have ${f_1}_{E_2}h \succsim {g_1}_{E_2}h$. Suppose $f_1 \succ_1 g_1$. E_2 being non-null there exist $f, g, k \in \mathcal{F}$ such that $f_{E_2}k \succ g_{E_2}k$. Consider $f_1^*, g_1^* \in \mathcal{F}_1$ such that $f_1^* \succsim_1 f(\cdot, s_2)$ and $g(\cdot, s_2) \succsim_1 g_1^*$ for all $s_2 \in \mathcal{S}_2$. The existence of such elements $f_1^*, g_1^* \in \mathcal{F}_1$ is guaranteed by the finiteness of the ranges of $\Phi(f)$ and $\Phi(g)$. By contrapositive of A3(i) there must be some $s_2 \in \mathcal{S}_2$ such that $f_{E_2}k(\cdot, s_2) \succ_1 g_{E_2}k(\cdot, s_2)$ and therefore $f(\cdot, s_2) \succ_1 g(\cdot, s_2)$. It follows that $f_1^* \succ_1 g_1^*$. Iterative applications of A3(i) yields

$${f_1^*}_{E_2}k \succsim f_{E_2}k \quad \text{and} \quad g_{E_2}k \succsim {g_1^*}_{E_2}k.$$

Since, $f_{E_2}k \succ g_{E_2}k$, we then have, by transitivity of $\succsim$ and A2

$${f_1^*}_{E_2}g_1^* \succ {g_1^*}_{E_2}g_1^* = g_1^* = {f_1^*}_{\emptyset}g_1^*.$$

Finally, by A4

$${f_1}_{E_2}g_1 \succ {f_1}_{\emptyset}g_1 = {g_1}_{E_2}g_1,$$

and by A2 again

$${f_1}_{E_2}h \succ {g_1}_{E_2}h.$$

□

Lemma 8 *Suppose $\succsim$ satisfies Axioms A1-A7. Let P_2 be the probability measure from Lemma 7. Then,*

(i) *For all $E_1 \subseteq \mathcal{S}_1$, there exists $E_2 \subseteq \mathcal{S}_2$ such that $E_1 \sim_\ell E_2$,*

(ii) *For all* $E_1 \subseteq \mathcal{S}_1$, $E_2 \subseteq \mathcal{S}_2$ *and all* $x^*, x, y^*, y \in \mathcal{X}$ *such that* $x^* \succ x$ *and* $y^* \succ y$, $x^*_{E_1}x \sim x^*_{E_2}x$ *if and only if* $y^*_{E_1}y \sim y^*_{E_2}y$.

(iii) *For all* $E_2, F_2 \subseteq \mathcal{S}_2$, $E_2 \succsim_\ell F_2$ *if and only if* $P_2(E_2) \geq P_2(F_2)$,

(iv) *For all* $E_1, F_1 \subseteq \mathcal{S}_1$ *and all* $x^*, x, y^*, y \in \mathcal{X}$ *such that* $x^* \succ x$ *and* $y^* \succ y$, $x^*_{E_1}x \sim x^*_{F_1}x$ *if and only if* $y^*_{E_1}y \sim y^*_{F_1}y$.

Proof. Consider $E_1 \subseteq \mathcal{S}_1$ and $x, y \in \mathcal{X}$ such that $x \succ y$ (existence guaranteed by A5). By A3 (ii), we obtain $x \succsim_1 x_{E_1}y \succsim_1 y$. Formula (3) further yields $U_1(x) \geq U_1(x_{E_1}y) \geq U_1(y)$. So there exists $\lambda \in [0,1]$ such that $U_1(x_{E_1}y) = \lambda \cdot U_1(x) + (1-\lambda) \cdot U_1(y)$. P_2 being convex-ranged, there exists a subset $E_2 \subseteq \mathcal{S}_2$ such that $P_2(E_2) = \lambda$. Then a new application of Formula (3) gives $x_{E_2}y \sim x_{E_1}y$ and therefore $E_1 \sim_\ell E_2$.

As for item (ii), consider $E_1 \subseteq \mathcal{S}_1$, $E_2 \subseteq \mathcal{S}_2$ and $x^*, x, y^*, y \in \mathcal{X}$ such that $x^* \succ x$ and $y^* \succ y$. Suppose that $x^*_{E_1}x \sim x^*_{E_2}x$. Observe that $\{y^*_{E_1}y \succsim z\} \sim_\ell \{y^*_{E_2}y \succsim z\}$ for all $z \in \mathcal{X}$. Indeed, if $z \succ y^*$ then $\{y^*_{E_1}y \succsim z\} = \{y^*_{E_2}y \succsim z\} = \emptyset$; if $y^* \succsim z \succ y$ then $\{y^*_{E_1}y \succsim z\} = E_1$ and $\{y^*_{E_2}y \succsim z\} = E_2$, and we have $E_1 \sim_\ell E_2$ because $x^*_{E_1}x \sim_\ell x^*_{E_2}x$; finally, if $y \succsim z$, then $\{y^*_{E_1}y \succsim z\} = \{y^*_{E_2}y \succsim z\} = \mathcal{S}$. By Axiom A7, $y^*_{E_1}y \sim y^*_{E_2}y$.

For any $x, y \in \mathcal{X}$ and $E_2, F_2 \subseteq \mathcal{S}_2$, formula (3) yields

$$x \succsim y \quad \Leftrightarrow \quad U_1(x) \geq U_1(y)$$

$$x_{E_2}y \succsim_2 x_{F_2}y \quad \Leftrightarrow \quad [U_1(x) - U_1(y)] \cdot [P_2(E_2) - P_2(F_2)] \geq 0.$$

By A5 there exists $x^*, x \in \mathcal{X}$ satisfying $x^* \succ x$ and thus

$$x^*_{E_2}x \succsim_2 x^*_{F_2}x \quad \Leftrightarrow \quad P_2(E_2) \geq P_2(F_2).$$

Item (iii) then immediately follows.

Finally, consider $E_1, F_1 \subseteq \mathcal{S}_1$ and $x^*, x, y^*, y \in \mathcal{X}$ such that $x^* \succ x$ and $y^* \succ y$. Suppose that $x^*_{E_1}x \sim x^*_{F_1}x$. By item (i), there exists $E_2, F_2 \subseteq \mathcal{S}_2$ such that $E_1 \sim_\ell E_2$ and $F_1 \sim_\ell F_2$. By item (ii), $x^*_{E_1}x \sim x^*_{E_2}x$ and $x^*_{F_1}x \sim x^*_{F_2}x$ but also $y^*_{E_1}y \sim y^*_{E_2}y$ and $y^*_{F_1}y \sim y^*_{F_2}y$. By transitivity of $\succsim$ we have $x^*_{E_2}x \sim x^*_{F_2}x$. It follows from item (iii) that $y^*_{E_2}y \sim y^*_{F_2}y$. Applying transitivity of $\succsim$ again we get $y^*_{E_1}y \sim y^*_{F_1}y$. □

It easily follows from Lemma 8 (and Axiom A1) that $\sim_\ell$ is transitive on $2^{\mathcal{S}_1} \cup 2^{\mathcal{S}_2}$ and hence an equivalence relation. Then, if $E_1 \subseteq \mathcal{S}_1$ and $E_2, F_2 \subseteq \mathcal{S}_2$ are such that $E_1 \sim_\ell E_2$ and $E_1 \sim_\ell F_2$, we have $E_2 \sim_\ell F_2$ or by item (iii) of Lemma 8 $P_2(E_2) = P_2(F_2)$. It is meaningful then to define a function v_1 from $2^{\mathcal{S}_1}$ to $[0,1]$ by setting, for all $E_1 \subseteq \mathcal{S}_1$,

$$v_1(E_1) \;=\; P_2(E_2), \tag{4}$$

where $E_2 \subseteq \mathcal{S}_2$ is any subset such that $E_1 \sim_\ell E_2$. Such a subset exists by Lemma 8(i).

Lemma 9 *Suppose $\succsim$ satisfies Axioms A1-A7. Let P_2 be the probability measure from Lemma 7 and v_1 be as in Formula (4). Then,*

(i) *For all $E_1 \subseteq \mathcal{S}_1$ and $E_2 \subseteq \mathcal{S}_2$, $E_1 \sim_\ell E_2$ if and only if $P_2(E_2) = v_1(E_1)$,*

(ii) *v_1 is a capacity on $\mathcal{S}_1$.*

Proof. The forward implication of (i) holds by definition of v_1. Suppose now $E_1 \subseteq \mathcal{S}_1$ and $E_2 \subseteq \mathcal{S}_2$ are such that $P_2(E_2) = v_1(E_1)$. By Lemma 8(i), there exists $F_2 \subseteq \mathcal{S}_2$ such that $E_1 \sim_\ell F_2$. Then, by the definition of v_1, we have $P_2(F_2) = v_1(E_1)$ and therefore $P_2(E_2) = P_2(F_2)$. Lemma 8 (iii) yields $F_2 \sim_\ell E_2$. $\sim_\ell$ being transitive on $2^{\mathcal{S}_1} \cup 2^{\mathcal{S}_2}$, we must have $E_1 \sim_\ell E_2$.

Clearly, we have $\emptyset_{\mathcal{S}_1} \sim_\ell \emptyset_{\mathcal{S}_2}$ (because $\emptyset \times \mathcal{S}_2 \sim_\ell \mathcal{S}_1 \times \emptyset$). This implies $v_1(\emptyset) = P_2(\emptyset) = 0$. Likewise, we have $\mathcal{S}_1 \sim_\ell \mathcal{S}_2$. This implies $v_1(\mathcal{S}_1) = P_2(\mathcal{S}_2) = 1$.

Finally, consider $E_1, F_1 \subseteq \mathcal{S}_1$ such that $E_1 \subseteq F_1$. Fix $x^*, x \in \mathcal{X}$ such that $x^* \succ x$. A standard application of A3(ii) shows $x^*_{F_1}x \succsim x^*_{E_1}x$. By Lemma 8(i), there exist $E_2, F_2 \subseteq \mathcal{S}_2$ such that $E_1 \sim_\ell E_2$ and $F_1 \sim_\ell F_2$. Furthermore, by Lemma 8(ii), $x^*_{F_2}x \sim x^*_{F_1}x$ and $x^*_{E_2}x \sim x^*_{E_1}x$. Then, by transitivity of $\succsim$, $x^*_{F_2}x \succsim x^*_{E_2}x$ and, by the definition of v_1 and Lemma 8(iii), $v_1(E_1) = P_2(E_2) \leq P_2(F_2) = v_1(F_1)$. □

Lemma 10 *Suppose $\succsim$ satisfies Axioms A1-A7. Let (P_2, U_1, v_1) be as in Lemma 7 and Formula (4). Let u be the non-constant function from $\mathcal{X}$ to $\mathbb{R}$ defined by $u(x) = U_1(x)$ for all $x \in \mathcal{X}$. Then, $U_1(f_1) = \int_{\mathcal{S}_1} u \circ f_1 \, dv_1$ for all $f_1 \in \mathcal{F}_1$.*

Proof. Consider $f_1 \in \mathcal{F}_1$. There exists a finite partition $\{E_1^1, \ldots, E_1^N\}$ of $\mathcal{S}_1$ and a collection $\{x^1, \ldots, x^N\}$ satisfying $x^1 \succsim \ldots \succsim x^N$ such that f is equal to x^n on E_1^n for all $n \in [1 \ldots N]$. Set also $E_1^0 = \emptyset$.
P_2 being convex-ranged, we can find a finite partition $\{E_2^1, \ldots, E_2^N\}$ of $\mathcal{S}_2$ such that $P_2(E_2^n) = v_1(E_1^1 \cup \ldots \cup E_1^n) - v_1(E_1^1 \cup \ldots \cup E_1^{n-1})$ for all $n \in [1 \ldots N]$. Let $f_2 \in \mathcal{F}_2$ be the act equal to x^n on E_2^n for every $n \in [1 \ldots N]$. By construction and by Lemma 9(i), we have

$$P_2(\{f_2 \succsim_2 x\}) \;=\; v_1(\{f_1 \succsim_1 x\}) \text{ and hence } \{f_1 \succsim_1 x\} \sim_\ell \{f_2 \succsim_2 x\} \text{ for all } x \in \mathcal{X}.$$

Then, by A7, we obtain $f_1 \sim f_2$. Formula (3) further yields

$$\begin{aligned}
U_1(f_1) &= \mathbb{E}_{P_2}[u \circ f_2] \\
&= \sum_{n=1}^{N} u(x_n) \cdot P_2(E_2^n) \\
&= \sum_{n=1}^{N} u(x_n) \cdot [v_1(E_1^1 \cup \ldots \cup E_1^n) - v_1(E_1^1 \cup \ldots \cup E_1^{n-1})] \\
&= \int_{\mathcal{S}_1} u \circ f_1 \, dv_1.
\end{aligned}$$

□

Proof of Theorem 2.

The sufficiency of the axioms follows readily from previous Lemmata.

Suppose (v_1, P_2, u) provides the representation of $\succsim$ claimed in Theorem 2 and let V denote the resulting representing functional from $\mathcal{F}$ to $\mathbb{R}$. The necessity of A5 is an immediate consequence of the fact that u is non-constant. Defining a mapping U_1 from $\mathcal{F}_1$ to $\mathbb{R}$ by $U_1(f_1) = \int_{\mathcal{S}_1} u \circ f_1 dv_1$ for all $f_1 \in \mathcal{F}_1$, we obtain a SEU representation of $\succsim$ as the one of Formula (3). Then, the preference induced by V on $\Phi(\mathcal{F})$ satisfies P1, P2, P3, P4, and P6 of Savage (1954) which respectively reformulate as A1, A2, A3*(i) (from the proof of Lemma 7), A4, and A6 on $\succsim$. Since acts are finitely-valued and $\mathcal{B}$-measurable, A3(i) is actually equivalent to: for all $E_2 \subseteq \mathcal{S}_2$, all $f_1, g_1 \in \mathcal{F}_1$ and all $h \in \mathcal{F}$ if $f_1 \succsim_1 g_1$, then $f_{1E_2}h \succsim g_{1E_2}h$. That latter condition is obviously implied by A3*(i). A3(ii) follows from the monotonicity of the Choquet's integral. A7 follows from the definition of the Choquet's integral and from noting that the restriction of V to $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) is the Choquet (resp. Lebesgue) expected utility with respect to v_1 (resp. P_2).

As for the uniqueness, suppose (v_1, P_2, u) and (v_1', P_2', u') both provide a representation of $\succsim$. Then, consider the mappings U_1 and U_1' from $\mathcal{F}_1$ to $\mathbb{R}$ defined by $U_1(f_1) = \int_{\mathcal{S}_1} u \circ f dv_1$ and $U_1'(f_1) = \int_{\mathcal{S}_1} u' \circ f dv_1'$ for all $f_1 \in \mathcal{F}_1$. Then, both (P_2, U_1) and (P_2', U_1') provide an SEU representation of $\succsim$ as in Lemma 7. By uniqueness, we must have $P_2 = P_2'$ and $U_1' = aU_1 + b$ for some $a > 0$ and $b \in \mathbb{R}$. The latter implies $u' = au + b$ and $v_1 = v_1'$. □

Appendix B – Proof of other results

Preliminary Results

Consider two sets $\mathcal{E}$ and $\mathcal{E}'$ endowed with capacities v and v' respectively. Let $\mathcal{B}_{\mathcal{E}\times\mathcal{E}'}$ denote the Boolean algebra generated by the rectangles $E \times E'$ for $E \subseteq \mathcal{E}$ and $E' \subseteq \mathcal{E}'$. Let ζ be a finitely-valued $\mathcal{B}_{\mathcal{E}\times\mathcal{E}'}$-measurable function from $\mathcal{E} \times \mathcal{E}'$ to $\mathbb{R}$. We define two capacities $v \otimes v'$ and $v' \otimes v$ on $\mathcal{E} \times \mathcal{E}'$ by setting, for all $A \subseteq \mathcal{E} \times \mathcal{E}'$,

$$v \otimes v'(A) := \int_{\mathcal{E}} \int_{\mathcal{E}'} \mathbf{1}_A \, dv' dv \qquad v' \otimes v(A) := \int_{\mathcal{E}'} \int_{\mathcal{E}} \mathbf{1}_A \, dv dv'.$$

Lemma 11 *For all $A \in \mathcal{B}_{\mathcal{E}\times\mathcal{E}'}$, if $\mathbf{1}_A$ is slice-comonotonic, then $v \otimes v'(A) = v' \otimes v(A)$.*

Proof. The result is straightforward from the definitions if A is a rectangle. Suppose now that v and v' are additive. Each $A \in \mathcal{B}_{\mathcal{E}\times\mathcal{E}'}$ is a finite disjoint union of rectangles, and the result follows from the additivity of v and v'.

We consider now the general case. Fix $e' \in \mathcal{E}'$. The functions $e \mapsto 1_A(e, e')$ and $e \mapsto v'(A(e))$ are comonotonic. Indeed, suppose $1_A(e_1, e') < 1_A(e_2, e')$ and $v'(A(e_1)) > v'(A(e_2))$. Then, $(e_1, e') \notin A$ and $(e_2, e') \in A$. But furthermore, since $\mathbf{1}_A$ is slice-comonotonic, we have $A(e_1) \subseteq A(e_2)$ or $A(e_1) \supseteq A(e_2)$. By the monotonicity of

v', we obtain $A(e_1) \supseteq A(e_2)$. Since $e' \in A(e_2)$, we finally obtain $e' \in A(e_1)$, a contradiction. Then, by Schmeidler (1986), there exists a probability measure P on $\mathcal{E}$ such that

$$v \otimes v'(A) \;=\; \int_{\mathcal{E}} \int_{\mathcal{E}'} \mathbf{1}_A \; dv' dP \qquad\qquad v' \otimes v(A) \;=\; \int_{\mathcal{E}'} \int_{\mathcal{E}} \mathbf{1}_A \; dP dv'.$$

Now fix $e \in \mathcal{E}$. Proceeding as in the previous paragraph, we can show that the functions $e' \mapsto \mathbf{1}_A(e, e')$ and $e' \mapsto P(A(e'))$ are comonotonic. Then, still by Schmeidler (1986), there exists a probability measure P' on $\mathcal{E}'$ such that

$$v \otimes v'(A) \;=\; \int_{\mathcal{E}} \int_{\mathcal{E}'} \mathbf{1}_A \; dP' dP \qquad\qquad v' \otimes v(A) \;:=\; \int_{\mathcal{E}'} \int_{\mathcal{E}} \mathbf{1}_A \; dP dP'.$$

The result finally follows from the first paragraph of the proof. □

Lemma 11 formalizes Ghirardato (1997) Remark 1 and proves that $v \otimes v'$ and $v' \otimes v$ both satisfy Ghirardato's condition of *Fubini-Independence*. Ghirardato's Lemma 3 then immediately implies Lemma 12 below.

Lemma 12 *If ζ has comonotonic $\mathcal{E}'$-sections, then,*

$$\int_{\mathcal{E}} \int_{\mathcal{E}'} \zeta \; dv' dv \;=\; \int_{\mathcal{E} \times \mathcal{E}'} \zeta \; dv \otimes v' \;=\; \int_{\mathcal{E} \times \mathcal{E}'} \zeta \; dv' \otimes v.$$

Finally, we will also need the following two lemmata:

Lemma 13 *If v is additive, then*

$$\int_{\mathcal{E}} \int_{\mathcal{E}'} \zeta \; dv' dv \;=\; \int_{\mathcal{E} \times \mathcal{E}'} \zeta \; dv \otimes v'.$$

Proof.

$$\begin{aligned}
\int_{\mathcal{E}} \int_{\mathcal{E}'} \zeta dv' dv \;&=\; \int_{\mathcal{E}} \left[\int_{[min(\zeta), max(\zeta)]} v'(\zeta \geq t) \; dt + min(\zeta) \right] \; dv \\
&=\; \int_{[min(\zeta), max(\zeta)]} \int_{\mathcal{E}} v'(\zeta \geq t) \; dv dt + min(\zeta) \\
&=\; \int_{[min(\zeta), max(\zeta)]} \int_{\mathcal{E}} \int_{\mathcal{E}'} \mathbf{1}_{\{\zeta \geq t\}} \; dv' dv dt + min(\zeta) \\
&=\; \int_{[min(\zeta), max(\zeta)]} v \otimes v'(\zeta \geq t) \; dt + min(\zeta) \\
&=\; \int_{\mathcal{E} \times \mathcal{E}'} \zeta \; dv \otimes v',
\end{aligned}$$

where the second line is by the additivity of v and because the inner integral takes only a finite number of values. □

Lemma 14 *If v and v' are both additive, then*

$$\int_{\mathcal{E}}\int_{\mathcal{E}'} \zeta \; dv'dv \;=\; \int_{\mathcal{E}'}\int_{\mathcal{E}} \zeta \; dvdv' \;=\; \int_{\mathcal{E}\times\mathcal{E}'} \zeta \; dv' \otimes v \;=\; \int_{\mathcal{E}\times\mathcal{E}'} \zeta \; dv \otimes v'.$$

Proof. It is an immediate consequence of the fact that ζ is finitely-valued and $\mathcal{B}_{\mathcal{E}\times\mathcal{E}'}$-measurable. See also Marinacci (1997). □

Proof of Proposition 1

Consider v_1, P_2 and u as Theorem 2 and fix an act $f \in \mathcal{F}$. Applying a positive affine transformation if necessary, we may suppose (without loss of generality) $min(u \circ f) = 0$ and $max(u \circ f) = 1$. Let $\lambda_{[0,1]}$ (resp. λ) denote the Lesbegue measure on Borel subsets of $[0, 1]$ (resp $\mathbb{R}$) and $R(f)$ be the function defined as follows:

$$R(f) : \begin{cases} \mathcal{S}_1 \times [0,1] & \to \quad [0,1] \\ (s_1, t) & \mapsto \quad P_2(u \circ f(s_1, \cdot) \geq t). \end{cases}$$

We now provide different reformulations for $ECU(f)$, $CEU(f)$ and $\overline{CEU}(f)$.

First, note that

- $CEU(f) = \int_{\mathcal{S}_1} \int_{[0,1]} R(f) \; dtdv_1$,
- $\overline{CEU}(f) \;=\; \int_{[0,1]} \int_{\mathcal{S}_1} R(f)dv_1(s_1)dt$.

In parallel, observe that:

$$\begin{aligned} \overline{CEU}(f) \;&=\; \int_{[0,1]}\int_{\mathcal{S}_1} P_2(u \circ f(s_1, \cdot) \geq t) \; dv_1(s_1)dt \\ &=\; \int_{[0,1]}\int_{\mathcal{S}_1}\int_{\mathcal{S}_2} \mathbb{1}_{\{u \circ f(s_1,s_2) \geq t\}} \; dP_2(s_2)dv_1(s_1)dt \\ &=\; \int_{[0,1]} v_1 \otimes P_2(u \circ f \geq t) \; dt \\ &=\; \int_{\mathcal{S}} u \circ f \; dv_1 \otimes P_2. \end{aligned}$$

Finally, by applying Lemma 13 to the definition of $ECU(f)$ and the first rewriting of $\overline{CEU}(f)$ above, we obtain:

- $ECU(f) = \int_{\mathcal{S}} u \circ f \; dP_2 \otimes v_1$,
- $\overline{CEU}(f) = \int_{[0,1]\times\mathcal{S}_1} R(f) \; d\lambda_{[0,1]} \otimes v_1$.

The formulas of $ECU(f)$ and $\overline{CEU}(f)$ as Choquet integrals on the product space $\mathcal{S}_1 \times \mathcal{S}_2$ above prove that the two criteria are indeed particular cases of Sarin & Wakker (1992) as claimed in Section 3. Now, by Lemma 12, we have that

- If f has comonotonic $\mathcal{S}_1$-sections, then
$\int_{\mathcal{S}} u \circ f \ dP_2 \otimes v_1 = \int_{\mathcal{S}} u \circ f \ dv_1 \otimes P_2 = \int_{\mathcal{S}_2} \int_{\mathcal{S}_1} u \circ f \ dv_1 dP_2$,
- If f has comonotonic $\mathcal{S}_2$-sections, then
$\int_{\mathcal{S}} u \circ f \ dP_2 \otimes v_1 = \int_{\mathcal{S}} u \circ f \ dv_1 \otimes P_2 = \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} u \circ f \ dP_2 dv_1$,
- If for any pair $t, t' \in [0,1]$, the functions $R(f)(\cdot, t)$ and $R(f)(\cdot, t')$ from $\mathcal{S}_1$ to $[0,1]$ are comonotonic, then
$\int_{[0,1]\times\mathcal{S}_1} R(f) \ d\lambda_{[0,1]} \otimes v_1 = \int_{[0,1]\times\mathcal{S}_1} R(f) \ dv_1 \otimes \lambda_{[0,1]} = \int_{\mathcal{S}_1} \int_{[0,1]} R(f) \ d\lambda_{[0,1]} dv_1$.

The first two items of Proposition 1 immediately follow. The third one follows from the next and last lemma.

Lemma 15 Consider $t, t' \in [0,1]$. If $f(s_1, \cdot) \trianglerighteq f(s_1', \cdot)$ or $f(s_1', \cdot) \trianglerighteq f(s_1, \cdot)$ for all $s_1, s_1' \in \mathcal{S}_1$, then $R(f)(\cdot, t)$ and $R(f)(\cdot, t')$ are comonotonic.

Proof. Fix $s_1, s_1' \in \mathcal{S}_1$ and suppose without loss of generality that $f(s_1, \cdot) \trianglerighteq f(s_1', \cdot)$. By definition of stochastic dominance and representation of Theorem 2 we have that

$$P_2(u \circ f(s_1, \cdot) \geq t) \geq P_2(u \circ f(s_1', \cdot) \geq t)$$

$$P_2(u \circ f(s_1, \cdot) \geq t') \geq P_2(u \circ f(s_1', \cdot) \geq t').$$

Therefore,

$$[P_2(u \circ f(s_1, \cdot) \geq t) - P_2(u \circ f(s_1', \cdot) \geq t)] \cdot [P_2(u \circ f(s_1, \cdot) \geq t') - P_2(u \circ f(s_1', \cdot) \geq t')] \geq 0$$

or again

$$[R(f)(s_1, t) - R(f)(s_1', t)] \cdot [R(f)(s_1, t) - R(f)(s_1', t)] \geq 0$$

As $s_1, s_1' \in \mathcal{S}_1$ are arbitrary chosen, the previous equation holds for any $s_1, s_1' \in \mathcal{S}_1$ and the functions $R(f)(\cdot, t)$ and $R(f)(\cdot, t')$ are comonotonic. □

Proof of Proposition 3 and Corollary 4

Lemma 16 *For any $V, W \in \mathcal{V}$, the following statements are equivalent:*

- $\succsim^V$ *is more ambiguity averse than* $\succsim^W$,
- $V(f) \leq W(f)$ *for all* $f \in \mathcal{F}$.

Proof. The second item trivially implies the first one. To prove that $\succsim^V$ more ambiguity averse than $\succsim^W$ implies $W(f) \geq V(f)$ for all $f \in \mathcal{F}$, we proceed by countrapositive.

First, note that every function V in $\mathcal{V}$ is monotonic is the following sense: for all $f, g \in \mathcal{F}$, if $f(s) \succsim g(s)$ for all $s \in \mathcal{S}$, then $f \succsim^V g$. Suppose $W(f) < V(f)$ for some $f \in \mathcal{F}$. Acts being finitely-valued, u being non-constant and invoking the monotonicity of V and W we can find consequences $x^*, x \in \mathcal{X}$ such that $u(x^*) > u(x)$ and $u(x^*) \geq V(f) > W(f) \geq u(x)$. Applying a positive affine transformation if necessary, we may suppose (without loss of generality) $u(x^*) = 1$ and $u(x) = 0$. By convex-rangedness of P_2, there exists $E_2 \subseteq \mathcal{S}_2$ such that $P(E_2) \in (W(f), V(f))$. Observe that $P(E_2) = \mathbb{E}_{P_2}[u(x^*_{E_2}x)] = V(x^*_{E_2}x) = W(x^*_{E_2}x)$. Hence, $V(x^*_{E_2}x) \geq V(f)$ while $W(x^*_{E_2}x) < W(f)$. $\succsim^V$ is not more ambiguity averse than $\succsim^W$. □

Lemma 17 *The following statements are equivalent:*

- $Core(v_1) \neq \emptyset$,
- $\succsim$ *is ambiguity averse,*
- $\succsim^{\overline{CEU}}$ *is ambiguity averse,*
- $\succsim^{CEU}$ *is ambiguity averse.*

In the case of $\succsim^{CEU}$, the equivalence between ambiguity aversion and the non-emptiness of the core of v_1 is already proven in Ghirardato & Marinacci (2002).

Proof. First, suppose $Core(v_1) \neq \emptyset$ and fix $P_1 \in Core(v_1)$. By monotonicity of Lebesgue's integral and monotonicity of Choquet's integral with respect to the measure we have that, for all $g \in \mathcal{F}$, $\mathbb{E}_{P_1}[\mathbb{E}_{P_2}[u \circ g]] \geq CEU(g)$ and $\mathbb{E}_{P_2}[\mathbb{E}_{P_1}[u \circ g]] \geq ECU(g)$. It immediately follows from these inequalities, by Lemma 14 and Lemma 16, that $\succsim^{CEU}$ and $\succsim^{ECU}$ are ambiguity averse. Now, consider two consequences $x^*, x \in \mathcal{X}$ such that $x^* \succ x$ (existence guaranteed by A5). Without loss of generality, we can set $u(x^*) = 1$ and $u(x) = 0$. Note that, for all $E \in \mathcal{B}$, $\mathbb{E}_{P_1}[\mathbb{E}_{P_2}[u(x^*_E x)]] = P_1 \otimes P_2(E)$ while $CEU(x^*_E x) = v_1 \otimes P_2(E)$. Thus, by first inequality above, $P_1 \otimes P_2(E) \geq v_1 \otimes P_2(E)$ for all $E \in \mathcal{B}$. We have seen in the proof of Proposition 1 that, $\overline{CEU}(g) = \int_{\mathcal{S}} u \circ g \; dv_1 \otimes P_2$ for all $g \in \mathcal{F}$. Hence, by monotonicity of Choquet's integral with respect to the measure, $\int_{\mathcal{S}} u \circ g \; dP_1 \otimes P_2 \geq \overline{CEU}(g)$. Finally, by Lemma 14 and Lemma 16, we have that $\succsim^{\overline{CEU}}$ is also ambiguity averse.

Now, suppose $\succsim$ (resp. $\succsim^{\overline{CEU}}$, resp. $\succsim^{CEU}$) is ambiguity averse or equivalently by Lemma 16 that there exists a probability measure P defined on $(\mathcal{S}, \mathcal{B})$ such that $\mathbb{E}_P[u \circ g] \geq ECU(g)$ (resp. $\mathbb{E}_P[u \circ g] \geq \overline{CEU}(g)$, resp. $\mathbb{E}_P[u \circ g] \geq CEU(g)$) for all $g \in \mathcal{F}$. Consider two consequences $x^*, x \in \mathcal{X}$ such that $x^* \succ x$ and suppose (without loss of generality) $u(x^*) = 1$ and $u(x) = 0$. Observe that for all $E_1 \subseteq \mathcal{S}_1$ we have $P(E_1 \times \mathcal{S}_2) = \mathbb{E}_P[u(x^*_{E_1}x)] \geq ECU(x^*_{E_1}x) = \overline{CEU}(x^*_{E_1}x) = v_1(E_1) = CEU(x^*_{E_1}x)$. Hence, the $\mathcal{S}_1$-marginal of P belongs to $Core(v_1)$. □

Lemma 18 *If $\succsim$ is ambiguity averse, then, $\succsim$ satisfies A8* (and thus A8).*

Proof. Suppose $\succsim$ is ambiguity averse. By Lemma 17, there exists $P_1 \in Core(v_1)$ and as shown in the proof of this Lemma, it satisfies, $\mathbb{E}_{P_2}[\mathbb{E}_{P_1}[u \circ g]] \geq ECU(g)$ or equivalently by Lemma 14, $\mathbb{E}_{P_1}[\mathbb{E}_{P_2}[u \circ g]] \geq ECU(g)$ for all $g \in \mathcal{F}$. Consider $f_2 \in \mathcal{F}_2$ and $g \in \mathcal{F}$ satisfying $f_2 \succsim_2 g(s_1, \cdot)$ for all $s_1 \in \mathcal{S}_1$. By the representation of Theorem 2 $ECU(f_2) \geq ECU(g(s_1, \cdot))$ for all $s_1 \in \mathcal{S}_1$. Observe that $ECU(g(s_1, \cdot)) = \mathbb{E}_{P_2}[u \circ g(s_1, \cdot)]$ and therefore $ECU(f_2) \geq \mathbb{E}_{P_1}[\mathbb{E}_{P_2}[u \circ g]]$. Hence, by hypothesis, $ECU(f_2) \geq ECU(g)$ or by the representation of Theorem 2 again, $f_2 \succsim g$. □

Lemma 19 *If $\succsim$ satisfies A8, then, $Core(v_1) \neq \emptyset$.*

Proof. Consider a finite family $\{E_1^i\}_{i=1}^N$ of events in $2^{\mathcal{S}_1}$ and a finite sequence of real numbers $\{\alpha^i\}_{i=1}^N \subseteq [0,1]$ such that $\sum_{i=1}^N \alpha^i \cdot \mathbb{1}_{E_1^i} = \mathbb{1}_{\mathcal{S}_1}$. Note that it must be $\alpha^1 + \ldots + \alpha^N \geq 1$. Since P_2 is convex-ranged, we can construct a partition $\{E_2^i\}_{i=1}^N$ of $\mathcal{S}_2$ satisfying, for all $i \in \{1, \ldots, N\}$,

$$P_2(E_2^i) := \frac{\alpha^i}{\alpha^1 + \ldots + \alpha^N}.$$

Since $\alpha^1 + \ldots + \alpha^N \geq 1$, we can also consider an event $E_2 \subseteq \mathcal{S}_2$ satisfying,

$$P_2(E_2) := \frac{1}{\alpha^1 + \ldots + \alpha^N}.$$

Now fix $x^*, x \in \mathcal{X}$ such that $u(x^*) > u(x)$ (existence guaranteed since u is non-constant). Applying positive affine transformation if necessary, we may suppose (without loss of generality) $u(x^*) = \alpha^1 + \ldots + \alpha^N$ and $u(x) = 0$. We define two acts $f_2 \in \mathcal{F}_2$ and $g \in \mathcal{F}$ as follows:

$$g(s_1, s_2) := \begin{cases} x^* & \text{if } (s_1, s_2) \in E_1^i \times E_2^i \text{ for some } i \in \{1, \ldots, N\}, \\ x & \text{otherwise.} \end{cases}$$

$$f_2(s_1, s_2) := \begin{cases} x^* & \text{if } s_2 \in E_2, \\ x & \text{otherwise.} \end{cases}$$

By definition, for any $s_1 \in \mathcal{S}_1$,

$$P_2(g(s_1, \cdot) = x^*) = \frac{\sum_{i=1}^N \alpha^i \cdot \mathbb{1}_{E_1^i}(s_1)}{\sum_{i=1}^N \alpha^i} = \frac{\mathbb{1}_{\mathcal{S}_1}(s_1)}{\sum_{i=1}^N \alpha^i} = \frac{1}{\sum_{i=1}^N \alpha^i} = P_2(f_2 = x^*),$$

and then necessarily

$$P_2(g(s_1, \cdot) = x) = P_2(f_2 = x).$$

In parallel,

$$ECU(g) = \sum_{i=1}^N \alpha^i \cdot v_1(E_1^i) \quad \text{and} \quad ECU(f_2) = \mathbb{E}_{P_2}[u \circ f_2] = 1.$$

Then, by A8, $f_2 \succsim g$, and by the representation of Theorem 2,

$$\sum_{i=1}^N \alpha^i \cdot v_1(E_1^i) \leq 1 = v_1(\mathcal{S}_1).$$

Finally, applying Schmeidler's (1967) extension of Bondareva-Shapley's theorem we get $Core(v_1) \neq \emptyset$. □

Proof of Proposition 5

Lemma 20 If v_1 is convex, then, $ECU(f) \leq \overline{CEU}(f) \leq CEU(f)$ for all $f \in \mathcal{F}$.

Proof. Fix an act $f \in \mathcal{F}$ and suppose (without loss of generality) $max(u \circ f) = 1$ and $min(u \circ f) = 0$.. As we have seen in the proof of Proposition 1, the representations ECU, $\overline{CEU}$ and CEU can be reformulated in several ways. Developing further some of them we obtain:

$$ECU(f) = \int_{\mathcal{S}} u \circ f \; dP_2 \otimes v_1 = \int_{[0,1]} \int_{\mathcal{S}_2} \int_{\mathcal{S}_1} \mathbb{1}_{\{u \circ f \geq t\}} \; dv_1 dP_2 dt,$$

$$\overline{CEU}(f) = \int_{\mathcal{S}} u \circ f \; dv_1 \otimes P_2 = \int_{[0,1]} \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} \mathbb{1}_{\{u \circ f \geq t\}} \; dP_2 dv_1 dt,$$

$$CEU(f) = \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} u \circ f \; dP_2 dv_1 = \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} \int_{[0,1]} \mathbb{1}_{\{u \circ f \geq t\}} \; dt dP_2 dv_1.$$

Now, suppose v_1 convex. By Schmeidler (1986),

$$\begin{aligned}
& \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} \int_{[0,1]} \mathbb{1}_{\{u \circ f \geq t\}} \; dt dP_2 dv_1 \\
= \; & min_{P_1 \in Core(v_1)} \left\{ \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} \int_{[0,1]} \mathbb{1}_{\{u \circ f \geq t\}} \; dt dP_2 dP_1 \right\} \\
= \; & min_{P_1 \in Core(v_1)} \left\{ \int_{[0,1]} \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} \mathbb{1}_{\{u \circ f \geq t\}} \; dP_2 dP_1 dt \right\} \\
\geq \; & \int_{[0,1]} min_{P_1 \in Core(v_1)} \left\{ \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} \mathbb{1}_{\{u \circ f \geq t\}} \; dP_2 dP_1 \right\} dt \\
= \; & \int_{[0,1]} min_{P_1 \in Core(v_1)} \left\{ \int_{\mathcal{S}_2} \int_{\mathcal{S}_1} \mathbb{1}_{\{u \circ f \geq t\}} \; dP_1 dP_2 \right\} dt \\
\geq \; & \int_{[0,1]} \int_{\mathcal{S}_2} min_{P_1 \in Core(v_1)} \left\{ \int_{\mathcal{S}_1} \mathbb{1}_{\{u \circ f \geq t\}} \; dP_1 \right\} dP_2 dt.
\end{aligned}$$

All the changes in the order of integration are here justified since f is finitely-valued and defined on the algebra generated by the rectangles. The triple inequality $ECU(f) \leq \overline{CEU}(f) \leq CEU(f)$ follows from the third and the fifth inequalities above. Indeed, by Schmeidler (1986) again we have:

$$\int_{[0,1]} min_{P_1 \in Core(v_1)} \left\{ \int_{\mathcal{S}_1} \int_{\mathcal{S}_2} \mathbb{1}_{\{u \circ f \geq t\}} \; dP_2 dP_1 \right\} dt \; = \; \overline{CEU}(f)$$

$$\int_{[0,1]} \int_{\mathcal{S}_2} min_{P_1 \in Core(v_1)} \left\{ \int_{\mathcal{S}_1} \mathbb{1}_{\{u \circ f \geq t\}} \; dP_1 \right\} dP_2 dt \; = \; ECU(f).$$

□

The next Lemma immediately follows from Lemma 16.

Lemma 21 *If $ECU(f) \leq \overline{CEU}(f) \leq CEU(f)$ for all $f \in \mathcal{F}$, then,*

- *$\succsim$ is more ambiguity averse than $\succsim^{\overline{CEU}}$,*
- *$\succsim^{\overline{CEU}}$ is more ambiguity averse than $\succsim^{CEU}$,*
- *$\succsim$ is more ambiguity averse than $\succsim^{CEU}$.*

Lemma 22 If $\succsim$ is more ambiguity averse than $\succsim^{\overline{CEU}}$ (respectively $\succsim^{CEU}$), then, $\succsim$ satisfies A9(i) (respectively A9(ii) and A9*).

Proof. Suppose $\succsim$ is more ambiguity averse than $\succsim^{\overline{CEU}}$. Consider $f, g \in \mathcal{F}$ such that $\{f(s_1, \cdot) = x\} \sim_\ell \{g(s_1, \cdot) = x\}$ for all $s_1 \in \mathcal{S}_1$ and $x \in \mathcal{X}$ and such that f has comonotonic $\mathcal{S}_1$-sections. Then, by Proposition 1, $ECU(f) = \overline{CEU}(f)$. However, since f and g induce by assumption the same AA act, we have $\overline{CEU}(f) = \overline{CEU}(g)$ and obtain $ECU(f) = \overline{CEU}(g)$. Finally, since $\succsim$ is more ambiguity averse than $\succsim^{\overline{CEU}}$, Lemma 16 yields $\overline{CEU}(g) \geq ECU(g)$, hence $ECU(f) \geq ECU(g)$ and $f \succsim g$. The proofs of A9(ii) and A9* are similar. □

Lemma 23 If $\succsim$ satisfies A9, then, v_1 is convex.

Proof. Now, suppose $\succsim$ satisfies A9. Consider $E_1, F_1 \subseteq \mathcal{S}_1$ and $x^*, x \in \mathcal{X}$ such that $u(x^*) > u(x)$ (existence guaranteed since u is non-constant). Applying positive affine transformations if necessary, we may suppose (without loss of generality) $u(x^*) = 2$ and $u(x) = 0$. P_2 being convex-ranged, there exists $E_2 \subseteq \mathcal{S}_2$ such that $P_2(E_2) = P_2(E_2^c) = 1/2$ or equivalently $E_2 \sim_\ell E_2^c$. Consider the two acts $f, g \in \mathcal{F}$ defined as follows:

$$f := (x^*_{E_1 \cup F_1} x)_{E_2} (x^*_{E_1 \cap F_1} x) \quad \text{and} \quad g := (x^*_{E_1} x)_{E_2} (x^*_{F_1} x)$$

Observe that f and g induce the same AA act and f is slice-comonotonic. Moreover, $ECU(f) = v_1(E_1 \cup F_1) + v_1(E_1 \cap F_1)$ and $ECU(g) = v_1(E_1) + v_1(F_1)$ Hence, applying A9 yields $f \succsim g$ or equivalently $v_1(E_1 \cup F_1) + v_1(E_1 \cap F_1) \geq v_1(E_1) + v_1(F_1)$. We have proven that v_1 is convex. □

Proposition 5 leaves an interesting question unanswered. Is it possible to compare CEU and $\overline{CEU}$ in terms of the ambiguity aversion they reveal? Lemmata 16 and 21 show that when v_1 is convex $\succsim^{\overline{CEU}}$ is necessarily more ambiguity averse than $\succsim^{CEU}$. However, the converse may fail to hold. For instance, we have $CEU = \overline{CEU}$ over all $\mathcal{F}$ whenever $\mathcal{X}$ has only two elements, even if v_1 is not convex.

Proof of Proposition 6

Axiom A10* is obviously necessary for v_1 to be additive. Moreover, Axiom A10* implies Axiom A10 which in turn implies that A9 but also the following "dual version" of A9 which captures an aversion to slice-comonotonicity:

(A9') For all $f, g \in \mathcal{F}$ with f slice-comonotonic, if $\{f(s_1, \cdot) = x\} \sim_\ell \{g(s_1, \cdot) = x\}$ for all $s_1 \in \mathcal{S}1$ and $x \in \mathcal{X}$, then, $f \precsim g$.

We can prove that A9' implies the concavity of v_1 in the same way we have proven that A9 implies the convexity of v_1. Finally, we have that A10 implies the additivity of v_1.

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